AdS/CFT: Emerging Islands and Wormholes

William Chan

Supervisor: Prof. Jerome Gauntlett

23.09.2022

Abstract

The AdS/CFT correspondence is a special example of a gauge-gravity duality between quantum field theories in d-dimensional spacetime and a gravity theory in $d+1$ spacetime dimensions with a d-dimensional boundary. Given this correspondence, we consider black holes in Anti-de Sitter spacetimes and how a new mechanism to reproduce the Page curve could manifest - a crucial aspect of the information paradox. To do this we outline the story of the entanglement entropy beginning with the Ryu-Takayanagi formula and some of its later generalisations: HRT, FLM, and quantum extremal surfaces. This will lead us to find emerging "islands" and notions of wormholes that are in the spirit of ER=EPR.

A dissertation submitted in fulfillment of the requirements for the degree of MSc in Quantum Fields and Fundamental Forces in the Theoretical Physics Group.

Acknowledgements

I would like to express my appreciation for Professor Jerome Gauntlett for being patient with me throughout the duration of this project, his guidance and insight has taught me a great deal both for this project and beyond that. To my brother Maxwell - without your support I would not have been able to get through the year. To my friends and colleagues Simran, Raymond, Joey, Neha, Mary, Cathy, Austin, and Charlie for your encouragement, intellectual discussion and enduring the proof-reading I sent your way.

Contents

Introduction

The objective of this project is to provide an outline and some examples of concepts of the gauge-gravity duality - specifically looking at the conjectured correspondence between conformal field theories (CFTs) and gravity theories on Anti-de Sitter (AdS) spacetime. When We delve into the nuances of entanglement entropy and the role it plays in exploring beyond the black hole horizon, giving rise to islands and wormholes. We explore more recent work in the field such as the notion of quantum extremal surfaces, the potential implications of this work beyond the information paradox, and future work that could be on the horizon when gazing in this particular direction.

The paper begins with an introduction to the aforementioned AdS/CFT correspondence, beginning with the key concepts and definitions of Anti-de Sitter spacetime and conformal field theories as we explore why this correspondence has gained so much traction - especially where studying black holes is concerned. Discussion of entanglement entropy, black holes in AdS and the information paradox is also included here and will be intrinsic to the consequent parts of the report.

We then delve into entanglement entropy and the derivation of the Page curve subsequent to winding this into the holographic duality as proposed by Ryu and Takayanagi so that a more recent formulation of this in the form of quantum extremal surfaces can be explored. We perform some of the calculations for holographic entanglement entropy to explicitly show some of the crucial results by Ryu and Takayanagi from their groundbreaking paper on the AdS/CFT correspondence.

Black holes and black hole thermodynamics are introduced in the latter parts of the discussion. We then shift the focus to black holes in AdS and perform some manipulations that allow us to use the AdS/CFT correspondence in spacetimes containing black holes. This then leads to discussion of the Hawking information paradox and the use of both quantum extremal surfaces and the notion of "islands" to provide a potential mechanism in which the Page curve can be reproduced.

Chapter 1

Conformal Field Theories

In order to introduce conformal field theories (CFTs) we must start with a more general and widely used quantum field theory (QFT). Obtaining a QFT begins with some classical field theory in which we have a some field $\phi(t, \vec{\boldsymbol{x}})$ which has some dynamics depending on degrees of freedom: time t and spatial components $\vec{\boldsymbol{x}}$. One can define the dynamics using a Lagrangian encoding the equations of motion. We start with a Lagrangian density $\mathcal{L}$, from which one can define the Lagrangian L and action S. We will consider theories here defined by a Lagrangian, allowing us to derive a Hamiltonian and conserved currents (among other quantities). These can then be quantised to bring to life a quantum field theory, introduced by Peskin and Schroeder [1] as the marriage of quantum mechanics (QM) and these field dynamics we have described, levelling up QM from considering particle systems to considering field systems.

1.1 Quantum Field Theory

1.1.1 Lorentz and Poincaré Group

Minkowski space in $d+1$-dimensions typically comprises of one time dimension and d spatial dimensions. We choose some coordinate frame $x^\mu = (t, \vec{\boldsymbol{x}})$ where $\vec{\boldsymbol{x}}$ is a spatial vector in d-dimensions. From this we define the Minkowski metric, with typical units in which $c = \hbar = 1$:

$$\eta_{\mu\nu} = \text{diag}(-1, +1, ..., +1), \tag{1.1}$$

where diag() denotes a matrix that is non-zero only on the leading diagonal, with the values of the leading diagonal given by the arguments. This metric defines flat spacetime and can also be written $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$.

On a given spacetime we can impose symmetries that reflect how the geometry we are in is expected to behave. For example, at a point in flat spacetime we expect that if we move ourselves by some amount (a translation) and stand at a new point, the physics here should be identical to where we stood previously. We formally define a translation by the following,

$$x^\mu \longrightarrow x^\mu + a^\mu, \tag{1.2}$$

for some constant spacetime vector a^μ.

We also expect that for a rotation or a boost our physics should not change either. The symmetry group of rotations and boosts is known as the Lorentz group $O(1,3)$, and we define the Lorentz transformations as

$$x^\mu \longrightarrow \Lambda^\mu_\nu x^\nu, \tag{1.3}$$

where the Λ matrices belong to $O(1,3)$. This means that they form a group under matrix multiplication such that the condition

$$\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha_\mu \Lambda^\beta_\nu, \tag{1.4}$$

is obeyed. We can also consider infinitesimal Lorentz transformations of the form $\Lambda^\mu_\nu = \delta^\mu_\nu + \epsilon v^\mu_\nu + O(\epsilon^2)$, meaning that $v_{\mu\nu}$ must be antisymmetric and its components represent boosts and rotation parameters - rapidity and angles respectively. We can express this infinitesimal transformation using a generator $M^{\alpha\beta}$ and the transformation parameters encoded in $\omega_{\mu\nu}$ as such

$$\Lambda^\mu_\nu = \delta^\mu_\nu - \frac{i}{2}(\omega_{\alpha\beta}M^{\alpha\beta})^\mu_\nu + ..., \tag{1.5}$$

where $M^{\alpha\beta} = -M^{\beta\alpha}$. We define the proper orthochronous Lorentz transformations as the group of Lorentz matrices within which $\det(\Lambda^\mu_\nu) = +1$, meaning that the transformation preserves time orientation. This group is denoted $SO^+(1,3)$ and has the form

$$\Lambda^\mu_\nu = (e^{-\frac{i}{2}(\omega_{\alpha\beta}M^{\alpha\beta})})^\mu_\nu, \tag{1.6}$$

making use of the exponential form of the matrix.
We take the generators to be,

$$(M^{\alpha\beta})^\mu_\nu = i\eta^{\mu\rho}(\delta^\alpha_\rho \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\rho), \tag{1.7}$$

which define the Lorentz algebra.
To form the Poincaré Group we want to combine both the translation and Lorentz algebra. Starting with translations, we know:

$$\bar{f}(x) = f(x-a) = f(x) - a^\mu \partial_\mu f(x) + ..., \tag{1.8}$$

Let the generator of translations be denoted

$$P_\mu = -i\partial_\mu : \qquad \bar{f}(x) = f(x) - ia^\mu P_\mu f(x) + ..., \tag{1.9}$$

where these P_μ obey the Abelian translation algebra

$$[P_\mu, P_\nu] = 0, \tag{1.10}$$

and we may write $\bar{f}(x) = e^{-ia^\mu P_\mu} f(x)$, or similarly for a vector field translation.
For the Lorentz algebra we start with the transformation $\bar{x} = \Lambda x$ such that

$$\bar{f}(x) = f(\Lambda^{-1}x), \qquad \bar{v}^\mu(x) = \Lambda^\mu_\nu v^\nu(\Lambda^{-1}x), \tag{1.11}$$

and we can rewrite the generator - for example, on a scalar field - using the translation generator P_μ as

$$M^{\alpha\beta}_{(scalar)} = x^\beta P^\alpha - x^\alpha P^\beta, \tag{1.12}$$

and similarly on a vector field as

$$(M^{\alpha\beta}_{(vector)})^\mu_\nu = (x^\beta P^\alpha - x^\alpha P^\beta)\delta^\mu_\nu + (M^{\alpha\beta})^\mu_\nu, \tag{1.13}$$

where the final term $(M^{\alpha\beta})^\mu_\nu$ is defined as in (1.7). Each of M, $M_{(scalar)}$, and $M_{(vector)}$ satisfy the antisymmetry condition on the generators as well as the Lorentz algebra:

$$[M^{\alpha\beta}_{(r)}, M^{\rho\sigma}_{(r)}] = (-i)\left(\eta^{\alpha\beta}M^{\beta\sigma}_{(r)} - \eta^{\beta\rho}M^{\alpha\sigma}_{(r)} - \eta^{\alpha\sigma}M^{\beta\rho}_{(r)} + \eta^{\beta\sigma}M^{\alpha\rho}_{(r)}\right), \tag{1.14}$$

with (r) simply indicating that this is a general statement. We bring both (1.10) and (1.14) define the Poincaré algebra together with the relation

$$[P^\mu, M^{\sigma\rho}_{(r)}] = i(\eta^{\mu\sigma}P^\rho - \eta^{\mu\rho}P^\sigma)\mathcal{I}_{(r)}, \tag{1.15}$$

Simply put, the Poincaré group is the union of the proper orthochronous Lorentz group and the translation group. In (1.15), $\mathcal{I}_{(r)}$ denotes the relevant identity matrix for the field theory under consideration.

1.2 Conformal Field Theories (CFTs)

A central theme throughout this report will be the concept of conformal invariance - a generalisation of Poincaré invariance discussed in the previous section such that, in addition to the theory being invariant under boosts, rotations and translations, we also have a scaling invariance of the field theory and special conformal transformations. This is significant as it allows for physics to be linked at several scales.

1.2.1 The Conformal Symmetry Group

Maldacena et al [2] introduce the conformal transformations as those preserving the form of the metric up to some scaling, that is to say $g_{\mu\nu}(x) \longrightarrow \Omega^2(x)g_{\mu\nu}(x)$, to form a group including the Poincaré group. Here we are using $g_{\mu\nu}(x)$ as some arbitrary localised metric.

We can consider this group in Minkowski space, breaking it down into its constituent transformations:

- The Poincaré transformations: Lorentz group (boosts, rotations) and the translation group.
- The scale transformations: $x^\mu \longrightarrow \lambda x^\mu$.
- The special conformal transformations: $x^\mu \longrightarrow \frac{x^\mu + a^\mu x^2}{1+2x^\nu a_\nu + a^2x^2}$.

Let's say we have some relativistic quantum field theory in d dimensions with the Minkowski metric $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ and we initially constrain the theory to have Poincaré invariance such that

$$\begin{aligned} x^\mu &\longrightarrow x^\mu + a^\mu, \\ x^\mu &\longrightarrow \Lambda^\mu_\nu x^\nu, \end{aligned} \tag{1.16}$$

where we now know $\Lambda \in SO(1, d-1)$.

Denoting our field ϕ, the action of the Poincaré generators is as follows:

$$\begin{aligned} [P_\mu, \phi] &= i\partial_\mu \phi, \\ [M_{\mu\nu}, \phi] &= (i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \mathcal{E}_{\mu\nu})\,\phi, \end{aligned} \tag{1.17}$$

where we have used $\mathcal{E}_{\mu\nu}$ to denote the intrinsic spin, schematically identical to the final term in (1.13) and some indices in ϕ have been suppressed. We note that the Casimir invariant of the Lorentz group is defined by $m^2 = -P_\mu P^\mu$, where this is just an element of the centre of the Lorentz group, however this is not a Casimir when scale symmetry is baked into the theory.

Now we introduce the scale symmetry (or dilation symmetry) in a similar way such that the transformation is $x^\mu \longrightarrow \lambda x^\mu$ with its action on some arbitrary function f being

$$Df = ix^\mu \partial_\mu f, \tag{1.18}$$

Here D can be taken as the generator of scale transformations and we find that the following relations hold:

$$\begin{aligned} [M_{\alpha\beta}, D] &= 0, \\ [P_\mu, D] &= iP_\mu, \end{aligned} \tag{1.19}$$

We now define the action of the scale generator on our field as

$$[D, \phi] = i(-\vartriangle + x^\mu \partial_\mu)\phi, \tag{1.20}$$

with an intrinsic scale $\vartriangle$ being introduced here. We note that (1.20) is the infinitesimal version of $\phi \longrightarrow \lambda^\vartriangle \phi(\lambda x)$, and in doing so denote $\vartriangle$ to be our *scale dimension*.

The generator for the special conformal transformations must also be defined and acts on the field as follows,

$$[K_\mu, \phi] = \left(i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \vartriangle) - 2x^\nu \mathcal{E}_{\mu\nu}\right)\phi. \tag{1.21}$$

We give the complete conformal algebra here:

$$\begin{aligned} &[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu); \quad [M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}P_\mu), \\ &[M_{\mu\nu}, M\rho\sigma_{(r)}] = (-i)\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho}\right); \\ &[M_{\mu\nu}, D] = 0; \quad [P_\mu, D] = iP_\mu; \quad [K_\mu, D] = -iK_\mu; \\ &[P_\mu, K_\nu] = 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D. \end{aligned} \tag{1.22}$$

keeping in mind that all other commutation relations vanish. The conformal algebra is isomorphic to the algebra of $SO(d,2)$ and can be defined via the algebra of $SO(d,2)$ generators that are linear combinations of the conformal algebra generators outlined above. These commutation relations actually suggest that P is an operator that acts to raise the dimension of the field, whereas in contrast, the operator K appears to act in such a way that the field dimension is lowered. For the field theory to be unitary, we must bound the dimension of fields from below, suggesting an operator of this minimal dimension exists in each representation of the conformal group that will be annihilated by K. This is the definition of a *primary operator (or field)* and we have defined their action on the field as operators already in (1.17), (1.20), and (1.21).

The representations of the conformal group can also be classified (see [2]) in terms of the maximal, compact subgroup $SO(d) \times SO(2)$, i.e. the $SO(d) \times SO(2)$ subgroup is not contained by any other proper subgroup of the conformal group, and it is compact when defined as a topological space. This will become more relevant in the radial quantisation of the CFT on the sphere $S^{d-1} \times \mathbb{R}$ which we will do in AdS space later on.

1.2.2 Correlation Functions

One of the fundamental objects we would like to construct in quantum field theories are correlation functions, these allow for the calculation of scattering amplitudes. Now that we are working with CFTs and the conformal group rather than the Poincaré group, because the former is much larger, our correlation functions of primary fields are much more restricted. Using the conformal algebra previously specified, one can show that the 2-point function for a scalar field is given by

$$\langle \phi(0)\phi(x) \rangle \propto \frac{1}{|x|^{2\triangle}} \equiv \frac{1}{(x^2)^{\triangle}}, \tag{1.23}$$

but vanishes when the fields are of different dimension. The 3-point function is constrained similarly as

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3) \rangle = \frac{c_{ijk}}{|x_1 - x_2|^{\triangle_1+\triangle_2-\triangle_3}|x_1 - x_3|^{\triangle_1+\triangle_3-\triangle_2}|x_2 - x_3|^{\triangle_2+\triangle_3-\triangle_1}}. \tag{1.24}$$

Maldacena et al [2] note that the correlation functions with energy-momentum tensor $T_{\mu\nu}$ can be related to correlation functions without T via the Ward identities of the conformal algebra. Necessarily, the dimension of the operator $T_{\mu\nu}$ must be $\triangle = d$, and similarly conserved currents of global symmetries must be operators such that $\triangle = d-1$. Because of unitarity, a lower bound is imposed on all field's dimensions - for example a scalar field obeys

$$\triangle \geq \frac{d-2}{2}, \tag{1.25}$$

with equality if and only if the theory is a free theory.

1.2.3 Operator Product Expansion (OPE)

Generally the OPE exists for any local field theory and is defined as taking two operators to a single point to create a local disturbance. Taking $O_1(x)$ and $O_2(y)$ as our operators, we write the OPE as:

$$O_1(x)O_2(y) \longrightarrow \sum_n C_{12}^n(x-y)O_n(y), \tag{1.26}$$

noting that this would typically be within a correlation function for it to have meaning. Heuristically, we are taking that local disturbance and expressing it as a sum containing all local operators acting at that point with the same quantum numbers as $O_1(x)O_2(y)$. For a CFT the OPE coefficient functions C_{12}^n can be determined by conformal invariance as

$$C_{12}^n = \frac{c_{12}^n}{|x-y|^{\Delta_1+\Delta_2-\Delta_n}}, \tag{1.27}$$

with c_{12}^n given by the 3-point correlation function defined previously (1.24).

It is important to then describe the radial quantisation of the theory, in which we choose time to be the radial direction in $\mathbb{R}^d$ and we take the origin to correspond to past infinity. This would imply that at any moment in fixed time our field theory lives on the sphere S^{d-1}. Now that we are on the sphere it is sensible to follow [2] and map our conformal algebra to $SO(d,2)$ and we can rewrite our generators (1.22) as $SO(d,2)$ generators J_{ab}:

$$J_{\mu\nu} = M_{\mu\nu}; \quad J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu); \quad J_{\mu(d+1)} = \frac{1}{2}(K_\mu - P_\mu); \quad J_{(d+1)d} = D, \tag{1.28}$$

allowing us to study the CFT in Euclidean space. Now the Hamiltonian for this radial quantisation is the operator $J_{0(d+1)} = \frac{1}{2}(K_\mu + P_\mu)$. We can then map an operator to a state via the relation

$$|O\rangle = \lim_{x\to 0} O(x)\,|0\rangle\,, \tag{1.29}$$

and we can essentially view the state as some ball with the operator O inserted at the origin corresponding to past infinity. We can perform the inverse by taking a state which is a functional of field values on some ball and then use conformal invariance to shrink the ball to a point coinciding with our origin. This is equivalent to inserting a local operator at the origin. We can then see that in the radial quantisation we have a *one-to-one correspondence* between local operators O and states $|O\rangle$.

1.2.4 Superconformal Algebras

We can consider another generalisation of the Poincaré algebra - the supersymmetry (SUSY) algebra. We want to combine supersymmetry with the conformal algebra to give a full superconformal algebra description. We add in fermionic operators Q and we include two more generator types:

- S: the fermionic generator (one per SUSY generator),

- R: R-symmetry generators that may appear to form some Lie algebra.

This can give the full superconformal algebra in addition to (1.22),

$$\begin{aligned} &[D, Q] = -\frac{i}{2}Q; \quad [D, S] = \frac{i}{2}S; \quad [K, Q] \simeq S; \quad [P, S] \simeq Q; \\ &\{Q, Q\} \simeq P; \quad \{S, S\} \simeq K; \quad \{Q, S\} \simeq M + D + R, \end{aligned} \tag{1.30}$$

with exact forms determined by the dimensions and exists only for $d \leq 6$, see [2].

For a free field without gravity we know that the maximum number of supercharges is 16, implying that the maximum number of fermionic generators in a field theory superconformal algebra is 32. These superconformal CFTs with the maximal amount of supercharges exist for $d = 3, 4, 6$ only;

$\boldsymbol{d = 3}$: the R-symmetry group is $Spin(8)$, with fermionic generators in $SO(3, 2) \times Spin(8)$,

$\boldsymbol{d = 4}$: the R-symmetry group is $SU(4)$, with fermionic generators in the $(\mathbf{4}, \mathbf{4}) + (\bar{\mathbf{4}}, \bar{\mathbf{4}})$ representation of $SO(4, 2) \times SU(4)$,

$\boldsymbol{d = 6}$: the R-symmetry group is $Sp(2) \simeq SO(5)$ with fermionic generators in the $(\mathbf{8}, \mathbf{4})$ representation of $SO(6, 2) \times Sp(2)$.

It is apparent that the conformal algebra is indeed a sub-algebra of the superconformal algebra, meaning that we can divide our representations of the superconformal algebra into representations of the conformal algebra. A primary field of the superconformal algebra (which will be annihilated at $x = 0$ by the K_μ and S generators) will include primary fields of the conformal algebra.

Superconformal algebras have special representations corresponding to *chiral primary operators* which are annihilated by combinations of the supercharges. These special representations are typically smaller than generic representations and contain fewer conformal primary fields, with the dimension of the chiral primary operators being determined uniquely by their R-symmetry representation such that, for a given representation, *dim*(non-chiral primary) > *dim*(chiral primary).

Chapter 2

Anti-de Sitter Spacetime

2.1 The Einstein Equations

We start with the Einstein equations including the cosmological constant Λ, which are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.1}$$

where $G_{\mu\nu} = \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$ is the Einstein tensor, $g_{\mu\nu}$ is the metric, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor. With a bit of work one can see that the Einstein equations come from the Einstein-Hilbert action adjusted for the cosmological constant:

$$S_{EH} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - 2\Lambda) + ..., \tag{2.2}$$

noting that g =det($g_{\mu\nu}$) and $\mathcal{M}$ is manifold we are concerned with. Before we can introduce the Anti-de Sitter spacetime we take a brief interlude into some of the context behind this. The cosmological geometry can be understood from the left hand side of the Einstein equations (2.1), and the matter content from the right hand side. Bringing these together we will better understand the dynamics of such gravitational systems.

2.1.1 Cosmological Geometry

Define our manifold to be $\mathcal{M} = \mathbb{R}(t) \times \Sigma(x^i)$, which can be thought of as a foliation with slices indexed by t. Taking Σ to be space-like and t to be time-like we state the general form of the ADM (Arnowitt, Deser, Misner) metric to be

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{2.3}$$

where N and N^i are the lapse and shift functions respectively, and h_{ij} is the induced or intrinsic metric on each of our foliation slices. Note that for convenience we will set $N^i = 0$ throughout as we expect our metric to be isotropic (which a vector field violates by definition). But we also expect homogeneity, telling us that N must only be a function of the time coordinate. We manipulate the metric given in (2.3) as follows

$$
\begin{aligned}
ds^2 &= (-N^2 + h_{ij}N^iN^j)dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j, \\
&= -N^2 dt^2 + h_{ij}dx^i dx^j, \\
&= -dt^2 + h_{ij}(t, x^i)dx^i dx^j,
\end{aligned}
\tag{2.4}
$$

where in the last line we absorb N into dt by redefining the time coordinate, whilst also making it explicit that the intrinsic metric is not restricted to depend solely on x^i. Whilst it is unclear what homogeneity implies for our intrinsic metric, we know that for a scalar, for example the Ricci scalar associated with h_{ij} should be uniform, that is to say:

$$
\begin{aligned}
R > 0 &\implies \text{Closed}, \\
R = 0 &\implies \text{Flat/Euclidean}, \\
R < 0 &\implies \text{Open},
\end{aligned}
\tag{2.5}
$$

where "closed", "flat" and "open" refer to the geometry. It is both convention and convenient to define a unit K such that

$$
\begin{aligned}
K = 1 &\implies \text{Closed}, \\
K = 0 &\implies \text{Flat/Euclidean}, \\
K = -1 &\implies \text{Open},
\end{aligned}
\tag{2.6}
$$

with $R = 6K$. Now the Ricci scalar may have some time dependence we can absorb this by taking $dx^i \longrightarrow a(t)dx^i$ such that

$$
h_{ij}(t, x^i) = a(t)^2\gamma_{ij}(x^i) \quad \implies \quad d\Sigma_f^2 = \gamma_{ij}dx^i dx^j. \tag{2.7}
$$

And so, we come to the definition of the flat FLRW (Friedman-Lemaître-Robertson-Walker, may also be seen as FRW) metric:

$$
ds^2 = dt^2 + a(t)^2 d\Sigma_f^2. \tag{2.8}
$$

2.1.2 Maximally Symmetric Spaces - Minkowski, de Sitter and Anti-de Sitter

For the space to be both homogeneous and isotropic we must define a Riemann tensor on our $d-1$ dimensional foliation slice R_{ijkl} that satisfies these constraints. Given the symmetries of the Riemann and the homogeneity and isotropy only allow for the metric to make up the tensorial structure, our only viable solution would be

$$
R_{ijkl} = C(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}), \tag{2.9}
$$

for C constant. By considering contractions, we see the implications for the Ricci scalar R:

$$
\begin{aligned}
\text{Ricci tensor: } R_{ik} &= R_{ijkl}\gamma^{jl} = C(3\gamma_{ik} - \delta_i^j\gamma_{jk}) = 2C\gamma_{ik}, \\
\text{Ricci scalar: } R &= R_{ik}\gamma^{ik} = 6C, \\
&\implies C = K.
\end{aligned}
\tag{2.10}
$$

We will work off of the loose definition that a maximally symmetric space can be specified by one obeying

$$R_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}). \tag{2.11}$$

We choose to define the full metric and intrinsic metric as such, with χ as the proper distance coordinate:

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2\gamma_{ij}dx^i dx^j, \\ &= -dt^2 + a(t)^2\left[d\chi^2 + f(\chi)^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right], \\ &\implies \gamma_{11} = 1, \quad \gamma_{22} = f(\chi)^2, \quad \gamma_{33} = \sin^2(\theta)f(\chi)^2, \end{aligned} \tag{2.12}$$

with all other values of γ_{ij} vanishing. Using our understanding of the Riemann tensor and (2.11) we can formulate a relationship between f and K. Begin with the Riemann but lowered and only using spatial coordinates, as well as the Christoffel symbols,

$$\begin{aligned} R_{ijkl} &= \partial_k\Gamma_{ilj} - \partial_l\Gamma_{ikj} + \Gamma_{ikm}\Gamma^m_{lj} - \Gamma_{ilm}\Gamma^m_{kj}, \\ \Gamma_{ijk} &= \frac{1}{2}\left(\partial_j\gamma_{ik} + \partial_k\gamma_{ij} - \partial_i\gamma_{jk}\right). \end{aligned} \tag{2.13}$$

We can calculate the non-zero Christoffel symbols (omitting the vanishing symbols and noting that we have lowered the usual “up” index to the first “down” index position) to get:

$$\begin{aligned} &\Gamma_{122} = -f(\chi)f'(\chi), \quad \Gamma_{133} = -\sin^2(\theta)f(\chi)f'(\chi), \\ &\Gamma_{212} = f(\chi)f'(\chi), \quad \Gamma_{233} = -\sin(\theta)\cos(\theta)f(\chi)^2, \\ &\Gamma_{313} = \sin^2(\theta)f(\chi)f'(\chi), \quad \Gamma_{323} = \sin(\theta)\cos(\theta)f(\chi)^2. \end{aligned} \tag{2.14}$$

We don’t include all of the details here but it is quite easy to show, given (2.14), that most of the equations specified by (2.11) will be trivial, however the non-trivial ones all give the same system of equations. We demonstrate this with one of the straightforward examples,

$$\begin{aligned} R_{1212} &= \partial_1\Gamma_{122} - \partial_2\Gamma_{112} + \Gamma_{11m}\Gamma^m_{22} - \Gamma_{12m}\Gamma^m_{12}, \\ &= \partial_1\Gamma_{122} - \Gamma_{122}\Gamma^2_{12}, \\ &= -\partial_\chi\left(f(\chi)f'(\chi)\right) - f(\chi)f'(\chi)\gamma^{2n}\Gamma_{n12}, \\ &= -f'(\chi)^2 - f(\chi)f''(\chi) + f(\chi)f'(\chi)\gamma^{22}\Gamma_{212}, \\ &= -f'(\chi)^2 - f(\chi)f''(\chi) + f'(\chi)^2, \\ &= -f(\chi)f''(\chi), \end{aligned} \tag{2.15}$$

leading us to the following differential equation for f and K:

$$\begin{aligned} &R_{1212} = -f(\chi)f''(\chi), \qquad R_{1212} = K(\gamma_{11}\gamma_{22}) = Kf(\chi)^2, \\ &\implies f''(\chi) + Kf(\chi) = 0. \end{aligned} \tag{2.16}$$

Now we are equipped to introduce some spaces; Minkowski that we know and love, but also de Sitter and Anti-de Sitter, the latter of which will be crucial to our discussion later. First we take our maximal symmetry solution (2.9) and enlarge this to include the entire space rather than an instance of foliation, before performing a similar manipulation:

$$
\begin{aligned}
R_{\mu\nu\rho\sigma} &= R_0(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \implies R_{\nu\sigma} = 3R_0 g_{\nu\sigma} \implies R = 12R_0, \\
\implies R_{\mu\nu\rho\sigma} &= \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}).
\end{aligned} \tag{2.17}
$$

We want to be able to evaluate this in a similar way to the above and so (given our enlargement to the entire spacetime) we redefine the metric to be of a suitable form, starting with the FRW metric from (2.12),

$$
ds^2 = -dt^2 + a(t)^2 \left[d\chi^2 + f(\chi)^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right]. \tag{2.18}
$$

We want to move to a coordinate system that permits an area, thus defining $f(\chi) = r$ to rewrite the metric. Begin with (2.16)

$$
\begin{aligned}
& f''(\chi) + Kf(\chi) = 0, \quad f(\chi) \longrightarrow r, \\
& \implies \frac{d^2r}{d\chi^2} + Kr = 0, \\
& \frac{d}{dr}\left(\frac{dr}{d\chi}\right)^2 + Kr = 0, \\
& \implies \frac{dr^2}{1 - Kr^2} = d\chi^2,
\end{aligned} \tag{2.19}
$$

where we have used the chain rule and solved as an ODE, rescaling for convenience where it is permitted. Our metric now reads

$$
ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega_2^2\right]. \tag{2.20}
$$

Now in a similar fashion to (2.14), (2.15) we may find similar equations to (2.16) but for our enlarged solution. We compute the (lowered, as before) Christoffel symbols,

$$
\begin{aligned}
& \Gamma_{011} = -\frac{a\dot{a}}{1 - Kr^2}, \quad \Gamma_{022} = -a\dot{a}r^2, \quad \Gamma_{033} = -a\dot{a}r^2\sin^2(\theta), \\
& \Gamma_{111} = -\frac{Kra^2}{(1 - Kr^2)^2}, \quad \Gamma_{122} = -a^2 r, \quad \Gamma_{133} = -a^2 r\sin^2(\theta), \quad \Gamma_{110} = \Gamma_{101} = \frac{a\dot{a}}{1 - Kr^2}, \\
& \Gamma_{233} = -a^2r^2\sin(\theta)\cos(\theta), \quad \Gamma_{212} = \Gamma_{221} = a^2 r, \quad \Gamma_{202} = \Gamma_{220} = a\dot{a}r^2, \\
& \Gamma_{313} = \Gamma_{331} = a^2 r\sin^2(\theta), \quad \Gamma_{323} = \Gamma_{332} = a^2r^2\sin(\theta)\cos(\theta), \quad \Gamma_{303} = \Gamma_{330} = a\dot{a}r^2\sin^2(\theta),
\end{aligned} \tag{2.21}
$$

where any vanishing symbols have been omitted. Following the same calculation as above we obtain

$$
\begin{aligned}
R_{1212} =&\partial_1\Gamma_{122} - \partial_2\Gamma_{112} + \Gamma_{11\mu}\Gamma^{\mu}_{22} - \Gamma_{12\mu}\Gamma^{\mu}_{12}, \\
=&\partial_1\Gamma_{122} + \Gamma_{110}g^{00}\Gamma_{022} + \Gamma_{111}g^{11}\Gamma_{122} - \Gamma_{122}g^{22}\Gamma_{212}, \\
=&\partial_r(-a^2r) + \frac{a\dot{a}}{1-Kr^2}(-1)(-a\dot{a}r^2), \\
&- \frac{Kra^2}{(1-Kr^2)^2}\left(\frac{1-Kr^2}{a^2}\right)(-a^2r) - (-a^2r)\left(\frac{1}{a^2r^2}\right)a^2r, \\
=&(\dot{a}^2+K)\left(\frac{a^2r^2}{1-Kr^2}\right).
\end{aligned}
\tag{2.22}
$$

But from (2.17) we can show the following

$$
\begin{aligned}
R_{1212} = R_0 g_{11}g_{22} = \frac{R_0 a^4 r^2}{1-Kr^2}, \\
\implies R_0 = \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}.
\end{aligned}
\tag{2.23}
$$

Due to the 0 component of the metric being in play this time around, we get an additional equation,

$$
\begin{aligned}
R_{0101} &= -\frac{a\ddot{a}}{1-Kr^2}, \\
R_{0101} &= R_0 g_{00}g_{11} = -\frac{R_0 a^2}{1-Kr^2}, \\
\implies R_0 &= \frac{\ddot{a}}{a},
\end{aligned}
\tag{2.24}
$$

using (2.17). It is worth noting that (2.23) is the only equation that arises from computing R_{jikl}, and that (2.24) is the only equation that arises from computing R_{0i0i} (other equations from $R_{\mu\nu\rho\sigma}$ are trivial).

Minkowski Spacetime, $R_0 = 0$:

For $R_0 = 0$, we have two cases for K's value, the first being $K = 0$:

$$
\left(\frac{\dot{a}}{a}\right)^2 = 0 \quad \implies \quad a(t) = a_0 \text{ constant } \forall t,
\tag{2.25}
$$

giving a constant expansion factor in the FLRW metric which can just be absorbed into (for example) a radial coordinate in $d\Sigma_f^2$ to obtain the usual flat space metric.

Our second case is $K = -1$, giving:

$$
\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} = 0 \quad \implies \quad a(t) = \pm t.
\tag{2.26}
$$

This appears to be curved space, but upon closer inspection we have intrinsic curvature within flat space due to our choice of foliation. We do not actually have extrinsic curvature here - one can show this via a coordinate transformation - however we have defined our

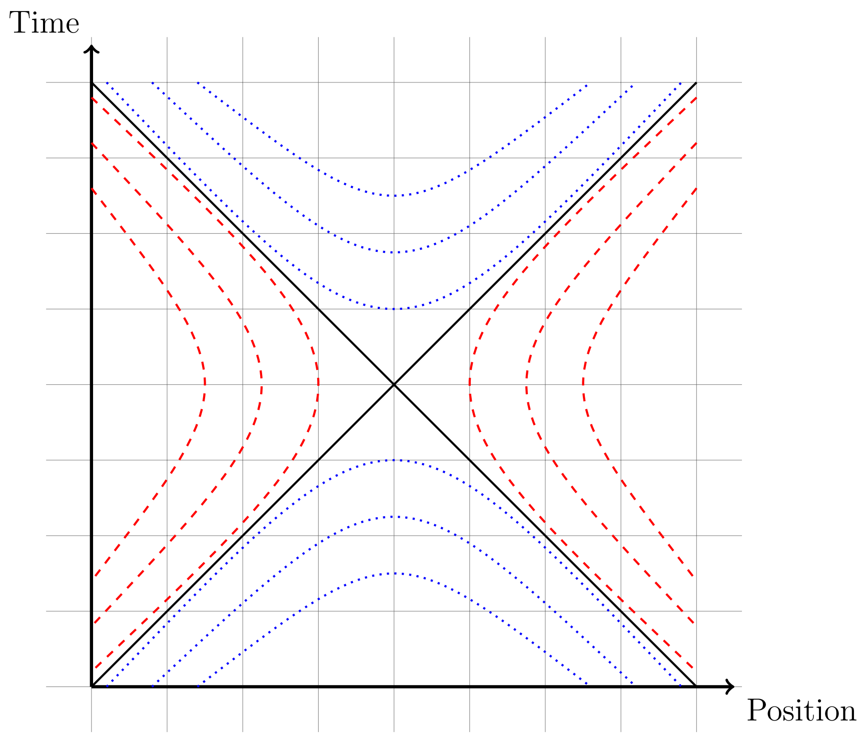

Figure 2.1: Minkowski space in curved coordinates where we've divided flat space into four wedges. The upper wedge is the expanding Milne universe with $a(t) = t$. Blue dotted lines indicate foliation slices. The lower wedge is the contracting Milne universe with $a(t) = -t$. The left and right wedges are the usual Rindler wedges. Red dashed lines indicate Rindler observer trajectories.

foliation slices in such a way that they are intrinsically curved. Diagrammatically, the $a = t$ corresponds to the upper wedge of a Rindler diagram (an expanding Milne universe), $a = -t$ corresponds to the lower wedge (contracting Milne universe), and the left and right wedges are the typical Rindler wedges (see Carroll [4]) as shown in Figure 2.1.

Without going into too much detail, we obtain the Rindler wedges through reversing the non-angular spatial and temporal coordinates from those used in the Milne universe wedges. Following the red dashed lines in Figure 2.1 maps out the trajectory of a Rindler observer (one with constant acceleration in their frame).

de Sitter (dS) Spacetime, $R_0 > 0$:

In this case, we can have FLRW foliations such that $K = 0, \pm 1$ (and more outside of FLRW foliations, but not to be discussed here). For $K = 0$:

$$\left(\frac{\dot{a}}{a}\right)^2 = R_0 \quad \Longrightarrow \quad a(t) = a_0 e^{\sqrt{R_0 t}}, \tag{2.27}$$

indicating exponential expansion and constant Hubble parameter $\frac{\dot{a}}{a} = R_0 = H(t)$. This is the steady state expanding model of the universe.

Let us also consider the $K = 1$ case:

$$\frac{\dot{a}^2 + 1}{a^2} = R_0 \quad \implies \quad a(t) = \frac{1}{\sqrt{R_0}} cosh(\sqrt{R_0} t). \tag{2.28}$$

From this we can rewrite the metric for de Sitter space as

$$ds^2 = -dt^2 + \frac{\cosh^2(\sqrt{R_0} t)}{R_0} \left(d\chi^2 + \sin^2(\chi) d\Omega_2^2 \right). \tag{2.29}$$

Anti-de Sitter (AdS), $R_0 < 0$:

We briefly give a view of AdS from an FLRW perspective before delving into the spacetime in the next section. In this case, only the $K = -1$ foliations are permitted in the FLRW representation (again, many others exist and some will be discussed presently). Within the current confines we have

$$\left(\frac{\dot{a}}{a} \right)^2 - \frac{1}{a^2} = -|R_0| \quad \implies \quad a(t) = \frac{\sin(\sqrt{-R_0} t)}{\sqrt{-R_0}}. \tag{2.30}$$

While it is true that we can get $a = 0$ periodically, this is merely a coordinate singularity and not an issue. We obtain the FRW metric for AdS:

$$ds^2 = -dt^2 + \frac{\sin^2(\sqrt{-R_0} t)}{R_0} \left(d\chi^2 + \sin^2(\chi) d\Omega_2^2 \right). \tag{2.31}$$

2.2 AdS Spacetime

Having given some cosmological background for the Anti-de Sitter space, namely in the context of the FRW metric, we now look to unpack the AdS geometry in much wider generality as we build towards the correspondence itself.

2.2.1 Conformal Compactification and AdS Geometry

One of the main tools that will make up this discussion is conformal compactification - one might recognise this from studying Penrose diagrams [38] - which essentially allows an infinite space to be mapped to a finite space. Take the Euclidean metric for example, which compactifies to the n-sphere S^n. An easy way to visualise this compactification would be to consider the projection of the sphere to flat space, where we compactify by assigning the singular point in the sphere (e.g. the "north pole") to spatial infinity.

We first present a very basic example of 2-dimensional Minkowski space described by the metric

$$ds^2 = -dt^2 + dx^2, \qquad (-\infty < t, x < \infty). \tag{2.32}$$

We first transform to lightcone coordinates $u_\pm = t \pm x$, before performing another transformation to conformally compactify:

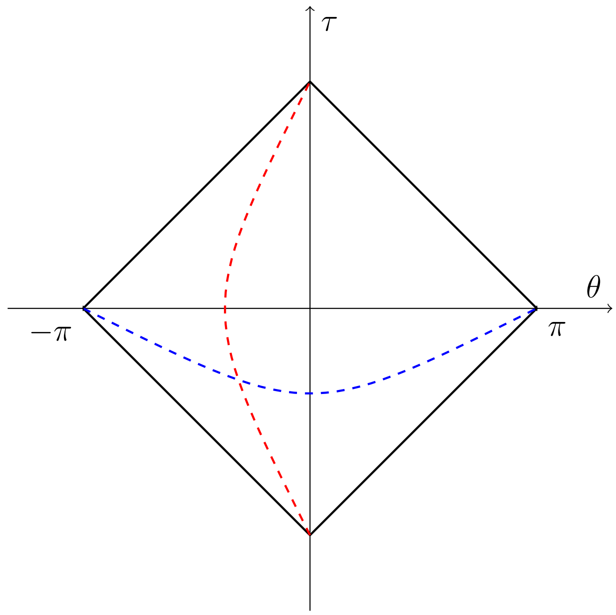

Figure 2.2: Conformally compactified Minkowski space in (τ, θ); the blue dashed line shows the trajectories of lines of constant t, the red dashed lines are lines of constant x.

$$\begin{aligned} ds^2 &= -du_+ du_-, \\ &= \frac{1}{4\cos^2(\tilde{u}_+)\cos^2(\tilde{u}_-)}(-d\tau^2 + d\theta^2), \end{aligned} \tag{2.33}$$

with the following redefinitions $u_\pm = \tan(\tilde{u}_\pm)$ where $\tilde{u}_\pm = \frac{(\tau+\theta)}{2}$. We can see that we have conformally mapped (2.32) to a metric defining a compact region $|\tilde{u}_\pm| < \frac{\pi}{2}$; the conformal scaling preserves any null trajectories and our coordinates (τ, θ) are well-defined at the aymptotic regions of flat space. We have *asymptotic flatness* - the spacetime has the same boundary structure as that of the original flat space after we perform conformal compactification.

In Figure 2.2 we can see the points that we've assigned to spatial infinity; $(\tau, \theta) = (0, \pm\theta)$ correspond to $x = \pm\infty$.

By identifying the points $\theta = \pm\pi$ one sees that we can essentially wrap our space to form a region of a cylinder, see Figure 2.3. It can be shown that correlation functions of a CFT on our $\mathbb{R}^{1,1}$ Minkowski can be analytically continued to the entire cylinder.

From this point, the same logic can be used to extend the above to higher dimensional Minkowski space $\mathbb{R}^{1,d}$, where $d \geq 2$. Following the procedure above as outlined by [2] we obtain the conformally compactified metric:

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2 d\Omega_{d-1}^2, \\ &= \frac{1}{4\cos^2(\tilde{u}_+)\cos^2(\tilde{u}_-)}(-d\tau^2 + d\theta^2 + \sin^2(\theta) d\Omega_{d-1}^2), \end{aligned} \tag{2.34}$$

where we have defined τ, θ, $u_\pm$ and $\tilde{u}_\pm$ exactly as in the $\mathbb{R}^{1,1}$ case. This allows us to define

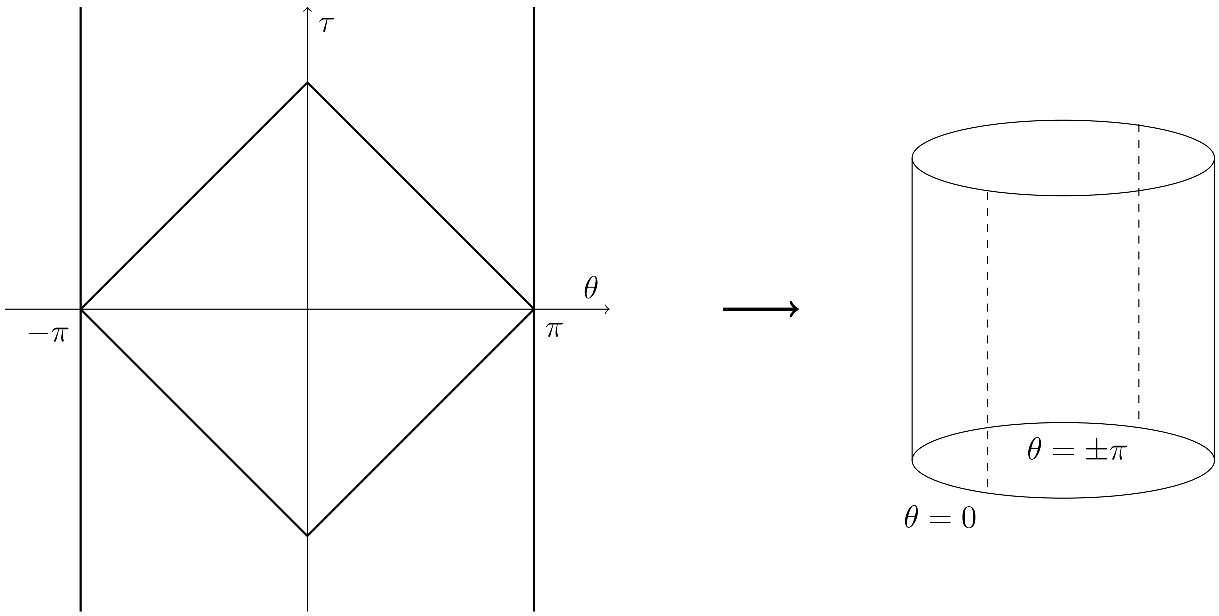

Figure 2.3: Identifying points $\theta = \pm\pi$ to "wrap" and analytically continue to a cylinder.

the conformally scaled metric

$$ds'^2 = -d\tau^2 + d\theta^2 + \sin^2(\theta) d\Omega_{d-1}^2. \tag{2.35}$$

This, as before, can be analytically continued to give the Einstein static universe geometry $\mathbb{R} \times S^d$. In doing so we have that $0 \leq \theta \leq \pi$ with $\theta = 0, \pi$ corresponding to S^d's north and south poles respectively, and $-\infty \leq \tau \leq +\infty$.

Taking a quick detour to consider the generators of the symmetry group in this case, one can see that

$$\partial_\tau = \frac{1}{2}(1 + u_+^2)\partial_{u_+} + \frac{1}{2}(1 + u_-^2)\partial_{u_-}, \tag{2.36}$$

which allows us to specify the generator of global time translations H to be identified with

$$H = \frac{1}{2}(P_0 + K_0) = J_{0,d+2} \tag{2.37}$$

The generators $J_{\mu\nu}$ are as defined in (1.28), $P_0 = \frac{1}{2}(\partial_{u_+} + \partial_{u_-})$ generates translations and $K_0 = \frac{1}{2}(u_+^2 \partial_{u_+} + u_-^2 \partial_{u_-})$ generates the special conformal transformations on $\mathbb{R}^{1,d}$. We know that our conformal symmetry group is $SO(2, d+1)$, and that H is the generator for the $SO(2)$ component of the maximally compact subgroup $SO(2) \times SO(d+1)$. Taking the universal cover of this subgroup we know saw earlier that it can be identified with the Einstein static universe's symmetry group. As pointed out in [2], the fact that H exists implies that one can always analytically extend correlation functions of a CFT on flat space $\mathbb{R}^{1,d}$ to the full Einstein static universe $\mathbb{R} \times S^d$.

We move swiftly on to present the anti-de Sitter space in $(d+2)$-dimensions, AdS_{d+2}, which we can write as a hyperboloid embedded in $(d+3)$-dimensional flat space. This set

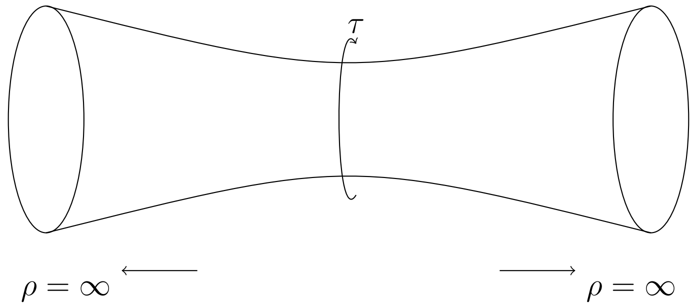

Figure 2.4: AdS_{d+2} hyperboloid embedded in $(d+3)$-dimensional flat space.

up has been constructed such that we have a space obeying homogeneity and isotropy, while possessing the $SO(2, d+1)$ isometry.

The hyperboloid for AdS_{d+2} is:

$$X_0^2 + X_{d+2}^2 - \sum_{i=1}^{d+1} X_i^2 = R^2. \tag{2.38}$$

We also define the flat space metric:

$$ds^2 = -dX_0^2 - dX_{d+2}^2 + \sum_{i=1}^{d+1} X_i^2. \tag{2.39}$$

Solving for the hyperboloid (2.38) we set

$$\begin{aligned} X_0 &= R\cosh(\rho)\cos(\tau), \\ X_{d+2} &= R\cosh(\rho)\sin(\tau), \\ X_i &= R\sinh(\rho)\Omega_i, \end{aligned} \tag{2.40}$$

in which $i = 1, ..., d+1$ and $\sum_i \Omega_i^2 = 1$. This allows us to rewrite the metric on AdS_{d+2} as

$$ds^2 = R^2(-\cosh^2(\rho)d\tau^2 + d\rho^2 + \sinh^2(\rho)d\Omega^2). \tag{2.41}$$

The coordinate ranges $\rho \geq 0$ and $0 \leq \tau \leq 2\pi$ covers the hyperboloid exactly once, and hence we may think of (τ, ρ, Ω_i) as the *global coordinates* of AdS space. In the limit as $\rho \longrightarrow 0$, this has the topology of $S^1 \times \mathbb{R}^{d+1}$, with S^1 representing closed time-like curves in τ. In this scenario the metric reduces to

$$ds^2 \simeq R^2(-d\tau^2 + d\rho^2 + \rho^2 d\Omega^2). \tag{2.42}$$

To obtain a universal covering of AdS without any closed time-like curves we can "unwrap" the hyperboloid in S^1 - we refer to this universal cover from here on as AdS_{d+2}. To see the conformal structure of AdS_{d+2} we introduce a coordinate $\theta : \quad \tan(\theta) = \sinh(\rho)$ with range $0 \leq \theta \leq \frac{\pi}{2}$, resulting in the metric

$$
\begin{aligned}
ds^2 &= \frac{R^2}{\cos^2(\theta)}(-d\tau^2 + d\theta^2 + \sin^2(\theta)d\Omega^2), \\
\implies ds^2 &= -d\tau^2 + d\theta^2 + \sin^2(\theta)d\Omega^2,
\end{aligned}
\tag{2.43}
$$

where, in the second line, we have used the scale invariance of causal structure. This final metric we have obtained is the Einstein static universe we saw in the previous section. More specifically, given the ranges of the coordinates we actually have a space-like hypersurface of constant τ corresponding to a $(d+1)$-dimensional hemisphere. The equator lies at $\theta = \frac{\pi}{2}$ and is the boundary with S^d topology.

Generally speaking, one has an *asymptotically AdS* spacetime if at the large spatial limits the metric approaches (2.43). To ensure we have a well-defined Cauchy problem on AdS we must specify boundary conditions at $\theta = \frac{\pi}{2}$. Essentially, we require that the boundary of our conformally compact AdS_{d+2} space is identical to the conformally compactified Minkowski space - crucial for the AdS/CFT correspondence.

We now move to the Poincaré chart as we will see later that some calculations will be easier to conceptualise and perform in these coordinates. Begin by first defining the coordinates

$$
\begin{aligned}
X_0 &= \frac{1}{2u}(1 + u^2(R^2 + \vec{x}^2 - t^2)), \\
X^i &= Rux^i, \qquad (i = 1, ..., p), \\
X^{d+1} &= \frac{1}{2u}(1 - u^2(R^2 - \vec{x}^2 + t^2)), \\
X_{d+2} &= Rut,
\end{aligned}
\tag{2.44}
$$

with $u, \vec{x} > 0$ and $u, \vec{x} \in \mathbb{R}^d$, covering one half of the hyperboloid (2.38). We then substitute this into the flat space metric (2.39) to find a new AdS_{d+2} metric:

$$
ds^2 = R^2 \left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2) \right). \tag{2.45}
$$

This is AdS in *Poincaré coordinates*, it is clear here that the subgroups $ISO(1,d)$ and $SO(1,1)$ of the AdS isometry group $SO(2,d+1)$ are manifest. The former encode the Poincaré transformations on $(t, \vec{x})$ and the latter being the tranformations $(t, \vec{x}, u) \longrightarrow (ct, c\vec{x}, c^{-1}u)$ with $c > 0$ a constant. The $SO(1,1)$ transformation we just outlined is identified with the dilation in the conformal symmetry group of $\mathbb{R}^{1,d}$ via the AdS/CFT correspondence.

For theories on AdS space with that have a positive definite Hamiltonian with respect to τ, the correlation functions of fields on the Euclidean space can be related to time-ordered correlation functions in Minkowski space via a *Wick rotation*. The metric of AdS_{d+2} is static with respect to our global coordinate τ, meaning that we can take $e^{i\tau H} \longrightarrow e^{-\tau_E H}$, where the Wick rotation is $\tau \longrightarrow \tau_E = -i\tau$. This redefines the hyperboloid given in (2.40) as

$$
\begin{aligned}
X_0^2 - X_E^2 - \vec{X}^2 &= R^2, \\
ds_E^2 &= -dX_0^2 + dX_E^2 + d\vec{X}^2,
\end{aligned}
\tag{2.46}
$$

where we have used $X_{d+2} \longrightarrow X_E = -iX_{d+2}$. In flat Minkowski space, the Euclidean rotation of t in the Rindler space $ds^2 = -r^2dt^2 + dr^2$ gives the flat Euclidean plane $\mathbb{R}^2$ despite the Rindler coordinates only covering a quarter (one wedge) of Minkowski $\mathbb{R}^{1,1}$. We can obtain the same space as (2.46) by rotating the time coordinate t of Poincaré coordinates (2.45) using $t \to t_E = it$ (despite this only covering half of the hyperboloid (2.38)).

Starting with the rotated global coordinate metric we can transform to Euclidean AdS_{d+2} space to better perform field theory computations;

$$\begin{aligned} ds_E^2 &= R^2(\cosh^2(\rho)d\tau_E^2 + d\rho^2 + \sinh^2(\rho)d\Omega_p^2), \\ &= R^2\left(\frac{du^2}{u^2} + u^2(dt_E^2 + d\vec{x}^2)\right), \\ &= \frac{R^2}{z^2}\left(dz^2 + dx_1^2 + ... + dx_{d+1}^2\right). \end{aligned} \tag{2.47}$$

This holds in AdS space if the theory has a positive-definite Hamiltonian with respect to the global coordinate τ; that is, correlation functions $\langle\phi_1...\phi_n\rangle$ of fields in Euclidean space when Wick rotated correspond to T-ordered correlation functions $\langle 0| T(\phi_1...\phi_n) |0\rangle$ in Minkowski space. For Euclidean AdS_{d+2} we essentially have a map to the $(d+2)$-dimensional disk. For example, in our $(u, t_E, \vec{x})$, $u = \infty$ is S^{d+1} at the boundary with a single point removed. For the full sphere we simply add the removed point back in and identify this with $u = 0 \quad (x = \infty)$. This is our compactification of Euclidean space to the Poincaré disk.

If we consider Euclidean AdS_2 this is the upper-half plane with the Poincaré metric and maps to the disk in such a way that the ∞ of the upper-half place correspondes to the boundary of the disk. If one were to instead take Lorentzian coordinates, $u = 0$ would represent a Killing horixon that acts as the boundary of our $(u, t, \vec{x})$ space. Note that in this case $u = 0$ is a null plane, but in the Euclidean case it is merely a single point.

2.2.2 Particles and Fields

Now that we have briefly introduced the geometry of the AdS spacetime, we want to explore the consequences of that geometry on the particles and fields residing within. There is actually quite a sizeable difference in the behaviour of massive particles on AdS geodesics compared to a massless particle like a photon. A massive particle for example will never reach the AdS boundary following a geodesic trajectory, whereas for a photon it can "bounce" back off of the boundary in finite time under the correct conditions.

Speaking with a bit more precision about the photon, consider the Penrose diagram of AdS - the cylinder we saw ealier (Figure 2.3), photons are able to reflect at the boundary (as seen by an observer on an AdS geodesic) with the appropriate boundary conditions set on the fields propagating in AdS.

Consider a scalar field propagating in our AdS_{d+2} space, the field equation and stationary wave solutions are given by:

$$\begin{aligned} &(\triangle - m^2)\phi = 0, \\ &\phi = e^{i\omega\tau}G(\theta)Y_l(\Omega_p). \end{aligned} \tag{2.48}$$

$G(\theta)$ is a hypergeometric function with θ defined as in (2.43), fully defined as:

$$
\begin{aligned}
G(\theta) &= \sin^l(\theta)\cos^{\lambda_\pm}(\theta)_2F_1(a,b,c;\sin(\theta)),\\
a &= \frac{1}{2}(l+\lambda_\pm-\omega R),\\
b &= \frac{1}{2}(l+\lambda_\pm+\omega R),\\
c &= l+\frac{1}{2}(d+1),\\
\lambda_\pm &= \frac{1}{2}(d+1)\pm\frac{1}{2}\sqrt{(d+1)^2+4(mR)^2}.
\end{aligned}
\tag{2.49}
$$

$Y_l(\Omega_d)$ is a spherical harmonic function and an eigenstate of the Laplacian on S^d with eigenvalue $l(l+d-1)$.

We can also define the *energy-momentum tensor*:

$$T_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}((\partial\phi)^2+m^2\phi^2)+\beta(g_{\mu\nu}\vartriangle - D_\mu D_\nu + R_{\mu\nu})\phi^2, \tag{2.50}$$

which is conserved for $\beta =$constant. β itself is determined by the choice of coupling of the scalar curvature to the field ϕ^2 (it has the same effect on AdS as the mass term in the wave equation) - this choice depends on the theory being considered for the scalar field.

The total energy of the scalar field fluctuation is

$$E = \int d^{d+1}x\sqrt{-g}T^0_0 \tag{2.51}$$

and is conserved if and only if the flux of energy-momentum through the boundary ($\theta = \frac{\pi}{2}$) is equal to zero,

$$\int_{S^d} d\Omega_d\sqrt{g}n_iT^i_0|_{\theta=\frac{\pi}{2}} = 0, \tag{2.52}$$

which can then be reduced to a boundary condition:

$$\tan^d(\theta)[(1-2\beta)\partial_\theta + 2\beta\tan(\theta)]\phi^2 = 0 \qquad (\theta \longrightarrow \frac{\pi}{2}). \tag{2.53}$$

This is only satisfied if for our stationary wave solutions, specifically in $G(\theta)$ (2.49), we have $a \in \mathbb{Z}$ or $b \in \mathbb{Z}$. Demanding the energy $\omega \in \mathbb{R}$ we find that

$$|\omega|R = \lambda_\pm + l + 2n \qquad (n = 0,1,2,...), \tag{2.54}$$

implying that we must have $\lambda \in \mathbb{R}$. We conclude from this that our mass is bounded from below such that,

$$m^2R^2 \geq -\frac{1}{4}(d+1)^2. \tag{2.55}$$

This is called the Breitenlohner bound and interestingly, this bound actually permits a (slightly) negative mass - sometimes this is referred to as being “slightly tachyonic”. The Compton wavelength for these possible tachyons is comparable to the curvature radius of AdS.

2.2.3 Brief Overview of Supersymmetry in AdS

We give a brief overview of supersymmetry in AdS following [2], from which we know that AdS_{d+2} has a *AdS supergroup* - a supersymmetry generalisation of the $SO(2, d+1)$ isometry group. For $d = 4$ and $\mathcal{N} = 1$ (the number of supercharges we have a simple supergravity theory:

$$S = \int d^4x \left(-\sqrt{g}(R - 2\Lambda) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu \gamma^5 \gamma_\nu \tilde{D}_\rho \psi_\sigma \right), \tag{2.56}$$

where we have the usual γ-matrices and Levi-Civita tensor, gravitino (spinor field) ψ, and standard covariant derivative D_μ which defines:

$$\tilde{D}_\mu = D_\mu + \frac{i}{2}\sqrt{\frac{\Lambda}{3}}\gamma_\mu. \tag{2.57}$$

We also define the supersymmetry variations of the gravitino and vierbein respectively as follows:

$$\begin{aligned} \delta\psi_\mu &= \tilde{D}_\mu \epsilon(x), \\ \delta V_{a\mu} &= -i\bar{\epsilon}(x)\gamma_a \psi_\mu. \end{aligned} \tag{2.58}$$

By requiring that the gravitino variation in annihilated $\delta\psi_\mu = 0$, we can determine the global supersymmetry of a given background. This is the only requirement we need as a bosonic configuration requires $\delta\psi_\mu = 0$, implying that the variation of the vierbein vanishes immediately. This results in the condition for $\epsilon(x)$

$$\tilde{D}_\mu \epsilon = \left(D_\mu + \frac{i}{2}\sqrt{\frac{\Lambda}{3}}\gamma_\mu \right) \epsilon = 0, \tag{2.59}$$

formally known as the *Killing spinor equation.* Note that to be integrable, one also requires that

$$[\tilde{D}_\mu, \tilde{D}_\nu] = \frac{1}{2}\left(R_{\mu\nu\rho\sigma}\sigma^{\rho\sigma} - \frac{2}{3}\Lambda\sigma_{\mu\nu} \right) \epsilon = 0, \qquad (\sigma_{\mu\nu} = \frac{1}{2}\gamma_{[\mu}\gamma_{\nu]}). \tag{2.60}$$

Due to AdS being a maximally symmetric space, we know the curvature obeys (in conjunction with the definition of the hyperboloid in the previous section)

$$R_{\mu\nu\rho\sigma} = \frac{1}{R^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{2.61}$$

By choosing $\Lambda = \frac{3}{R^2}$, one finds that (2.60) is satisfied for any ϵ. The Killing spinor equation is first order, meaning that there are as many solutions to the equation (2.59) as there are independent components of ϵ. This implies that AdS space preserves supersymmetry to the same extent that flat space does. With appropriate boundary conditions, our supergravity theory on AdS will be stable with energy bounded from below.

In theories where the number of supercharges are $\mathcal{N} > 1$ we have what is called an *extended supersymmetry.* For $d < 6$ these have been classified into AdS supergroups as follows by Nahm [5],

$$
\begin{aligned}
&AdS_4: \quad OSp(\mathcal{N}|4), \qquad \mathcal{N}=1,2,...,\\
&AdS_5: \quad SU(2,2|\mathcal{N}/2), \qquad \mathcal{N}=2,4,...,\\
&AdS_6: \quad F(4),\\
&AdS_7: \quad OSp(6,1|\mathcal{N}), \qquad \mathcal{N}=2,4.
\end{aligned}
\tag{2.62}
$$

AdS_{d+2} has no simple AdS supergroup for $d > 5$. Extended supersymmetries manifest as global symmetries of gauged supergravity on AdS_{d+2}. A crucial point of the AdS/CFT correspondence will be that these *extended supersymmetries are identified with the conformal superalgebras.*

Something to note but that won't be discussed here is that the gauged supergravities are supergravity theories with non-abelian gauge fields in the supermulitplet of the graviton. Typically $\Lambda < 0$, and AdS_{d+2} is a natural background geometry. Many are related to Kaluza-Klein compactifications of supergravities in $10-$ and $11-$dimensions.

We now take a deeper look into AdS_5 specifically, which has $\mathcal{N}=2,4,6,8$ gauged supergravities with supersymmetry $SU(2,2|\mathcal{N}/2)$.

Gauged $\mathcal{N}=8$ in AdS_5:

The gauge group is $SU(4) \simeq SO(6)$ with the global symmetry group being the E_6 group. One can derive this by truncating the compactification of the 10-dimensional type-IIB supergravity on S^5 via the Freund-Robin ansatz [39]. The Einstein equations imply that the self-dual 5-form field strength $F^{(5)}$ determines the radius of S^5 and cosmological constant $\Lambda = \frac{1}{R^2}$.

$\mathcal{N}=4$ in AdS_5:

The AdS/CFT correspondence conjectures that AdS_5 with $\mathcal{N}=4$ is dual to the large N limit of the $\mathcal{N}=4$ supersymmetric $SU(N)$ gauge theory in 4-dimensions.

The complete Kaluza-Klein mass spectrum of the type-IIB supergravity theory on $AdS_5 \times S^5$ is of particular interest as it quantises the frequency ω of stationary wave solutions, meaning that scalar field masses are of the form

$$
(mR)^2 = \tilde{l}(\tilde{l}+4),
\tag{2.63}
$$

with $\tilde{l}$ bounded from below. As $p=3$ in this case, we have that

$$
\begin{aligned}
\lambda_{\pm} &= 2 \pm |\tilde{l}+2|,\\
|\omega|R &= 2 \pm |\tilde{l}+2| + l + 2n, \qquad (n=0,1,2,...).
\end{aligned}
\tag{2.64}
$$

Hence we gather that all scalar fields in the supergravity multiplet are periodic in τ with period 2π, so before we take the universal cover of the hyperboloid, scalar fields are singled valued on the hyperboloid. The quantisation of ω is rooted in supersymmetry. Supergravity particles in 10-dimensions are BPS [17] and they preserve half of the supersymmetry. Having BPS particles in AdS supergravity like this has been shown to lead to this frequency quantisation in the case of 4-dimensions. Within the AdS/CFT correspondence this identifies with the notion that chiral operators have no anomalous dimensions.

Chapter 3

AdS/CFT: The Gauge-Gravity Duality

We have now outlined suffiecient prerequisites regarding conformal field theories (CFTs) and Anti-de Sitter (AdS) spacetime to begin talking about the AdS/CFT correspondence. We will begin with a brief introduction to the AdS/CFT correspondence itself - this "special case" of the gauge gravity duality - before taking a short interlude to discuss entanglement entropy. This will set us up to use the correspondence to make calculations of entanglement entropy using the procedure set out by Ryu and Takayanagi.

3.1 The Correspondence

Paraphrasing Maldacena [6] the gauge/gravity duality is a conjectured equality between a quantum field theory in d-dimensional spacetime and a gravity theory on a $d+1$-dimensional spacetime that has a d-dimensional asymptotic boundary. The AdS/CFT correspondence refers to a subset of examples of this that have been easiest to understand, and more specifically one writes this conjecture as:

Any gravity theory on $(d+1)$***-dimensional AdS spacetime with a*** d***-dimensional asymptotic boundary is dual to a conformal field theory in*** d***-dimensional spacetime***

We interpret the CFT to live on the boundary of the AdS space. The boundary of AdS is given by $\mathbb{R} \times S^{d-1}$ with the AdS isometry group behaving in such a way that points on the boundary are mapped to points on the boundary - this is just the action of the conformal group $SO(2, d)$. Consider the AdS metric as defined by (2.45). Taking $u = \frac{1}{z}$ we may rewrite the metric as such:

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2), \tag{3.1}$$

noting that $\vec{x}$ is a $(d-1)$-dimensional vector in the context as outlined here. This metric is invariant under rescaling, corresponding to a dilation symmetry on the boundary meaning we have a scale invariant boundary theory. Typically one can take this as an implication that the boundary theory is conformally invariant - crucially we note that this implies the energy-momentum tensor for this theory is traceless classically. Note that more generally we obtain the conformal anomaly for even dimensional theories.

Recalling that the boundary is $\mathbb{R} \times S^{d-1}$ and combining this with the knowledge that the boundary has conformal invariance, we set the radius of the boundary S^{d-1} to 1. The tracelessness of the energy-momentum tensor tells us that field theories with a metric $g_{\mu\nu}$ on the boundary is practically equivalent to a field theory with a boundary metric $\omega^2(x)g_{\mu\nu}$ up to a conformal anomaly.

To explore this correspondence and why it seems to hold between theories of dimension that differ by one, we consider counting the degrees of freedom of theories on both sides of the conjecture. In a large energy limit we consider the microcanonical ensemble and expect the entropy to behave as $S \propto V_{d-1}T^{d-1}$ for massless fields, where V denotes gravitational potential. We have a CFT on the boundary $\mathbb{R} \times S^{d-1}$ with the radius of S^{d-1} being 1, so for a temperature much greater than the boundary radius ($T \gg 1$) we expect

$$S \propto cT^{d-1}, \tag{3.2}$$

for c a dimensionless constant which Maldacena [6] notes as a measure of the effective number of fields a theory contains.

Now consider this from the point of view of the bulk - in our AdS and no longer at the boundary. Our theory on AdS appears to have massless particles (gravitons). Assuming for now that we have a theory with just gravitons we obtain a lower bound on the entropy possessed by this graviton gas:

$$S_{graviton} > T^d, \tag{3.3}$$

with the power law simply coming from the d spatial dimensions. Clearly this entropy is much greater than that of (3.2), but we must recall that the bulk theory is a gravitational theory containing black holes which put bounds on entropy. Consider the AdS black hole described by:

$$ds^2 = R^2 \left[-\left(r^2 + 1 - \frac{2gm}{r^{d-2}}\right) dt^2 + \left(r^2 + 1 - \frac{2gm}{r^{d-2}}\right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2 \right], \tag{3.4}$$

where m relates to the mass and g is defined with respect to Newton's constant

$$g \propto \frac{G_N^{(d+1)}}{R^{d-1}}. \tag{3.5}$$

The graviton gas extends up to $r \sim T$ with a mass $m \sim T^{d+1}$, which is roughly T^d in the large T-limit. Therefore we obtain for the Schwarzschild radius that

$$r_s^d \sim gm \sim gT^{d+1}. \tag{3.6}$$

From (3.4) we know that the Schwarzschild radius obeys $r_s^2 + 1 - \frac{2gm}{r_s^{d-2}} = 0$ as this gives our coordinate singularity at the horizon. A bit of rearranging, taking the large temperature limit and noting that $r_s \sim T$ will tell us that $r_s^d \sim gm$. We then simply apply $m \sim T^{d+1}$ to obtain the final part of (3.6).

Interestingly what this means is that the Schwarzschild radius is greater than the allowed size of the system for large temperatures $T > \frac{1}{g}$, invalidating (3.3). As the energy grows beyond a certain amount we must compute the entropy with respect to the black hole entropy which obeys an area law for the horizon $S \sim \frac{r_s^{d-1}}{g}$. For a large black hole we have $T_{Hawking} \propto r_s$. Therefore the black hole entropy is $S_{BH} \sim \frac{T^{d-1}}{g}$, which takes the form of (3.2) as we would have hoped, with

$$c \propto \frac{1}{g} \propto \frac{R^{d-1}}{G_N^{d+1}}. \tag{3.7}$$

This result is somewhat profound. The entropy of a black hole is connected with the thermal entropy of a field theory through the AdS/CFT correspondence. Maldacena [6] listed two important implications of this,

- Firstly, we have a statistical interpretation for the entropy of a black hole, and these black holes appear to adhere to quantum mechanics and unitarity. This is because we see the black hole here as a thermal state of a unitary QFT.
- Secondly, we are able to compute thermal properties in QFTs that have a dual gravity theory.

From (3.7) we also see that the number of degrees of freedom c scales inversely to the effective gravity coupling g, implying that to have a weakly coupled theory of gravity in the bulk one must have a large number of fields. The advantage of having a weakly coupled theory lies in the Fock space struture that comes with it, meaning one can discuss discrete particle states. Similarly in the dual CFT we see such a structure emerge in the large N limit.

For a gauge theory with large N that is based on gauge group $SU(N)$ with fields in the adjoint representation, one can find gauge invariant operators by tracing over fundamental fields. For example,

$$Tr(F_{\mu\nu}F^{\mu\nu}), \quad \text{or} \quad Tr(F_{\mu\nu}D_\rho D_\sigma F^{\mu\nu}). \tag{3.8}$$

Additionally one can have double trace operators that are just products of the above,

$$Tr(F_{\mu\nu}F^{\mu\nu}) \times Tr(F_{\mu\nu}D_\rho D_\sigma F^{\mu\nu}), \tag{3.9}$$

where acting with one on a vacuum of our QFT will create a state. For any CFT we know that there exists a map:

$$\mathbb{R} \times S^{d-1} \longrightarrow \mathbb{R}^d \tag{3.10}$$

in which our states on the cylinder (left hand side) are mapped to operators on the plane (right hand side) in such a way that the energy of a given state is equal to the dimension of the corresponding operator. One can see this as we saw earlier that the plane can be mapped to the Euclidean cylinder (Figure 2.3). This equivalence holds in any CFT.

As mentioned in Chapter One when discussing Operator Product Expansions (OPEs), such an operator at the origin creates a "disturbance" - this can be seen as creating a state at some radius r and thus a state of the field theory on the cylinder. In terms of the AdS/CFT correspondence, the equivalence is between such a state of the CFT on the cylinder and a state of the bulk theory in the global coordinates of AdS. As both have the same isometries, one can group states (operators) by how they transform in the conformal group. These representations are determined by both the operator's spin and scaling dimension.

Maldacena also refers to an argument made by 't Hooft [12] that taking the large N limit of a gauge theory results in a string theory. The argument allows one to organise diagrams in terms of those that can be drawn on topologies of increasing genus (a sphere, then a torus, etc). Increasing the genus increases the power of the $\frac{1}{N^2}$ coefficient of the Feynman diagrams, but one could interpret this as a string coupling with $g_s \sim \frac{1}{N}$. If one were to do this, it would quickly become apparent that the string theory in the bulk is exactly the string theory postulated by this argument, implying that in our weakly coupled gravitational theory we must also have the large N limit.

Gravitons have been discussed a few times now - these are simply the lowest string oscillation modes. In gravity, these gravitons are treated as a point particle and all massive string states of the theory are ignored, but to ignore the states in this way we require:

$$\frac{R_{AdS}}{l_s} \gg 1, \tag{3.11}$$

where l_s is the graviton size and R_{AdS} is the AdS radius. Maldacena [6] puts it as having a typical size of space that is much greater than the intrinsic graviton size means we have obtained a valid approximation of gravity. One can also relate this condition on gravitational coupling to the interaction strength between strings in the theory, namely that for the gravity approximation to be valid we must have a strongly interacting field theory.

3.1.1 Dynamics in AdS

To clarify further this relationship between operators on the boundary and fields within the bulk we consider the computation of the scalar field theory path integral, specifying appropriate fixed boundary conditions. Our understanding of quantum gravity right now really lies in the semiclassical and so we will take the semiclassical contribution, beginning with the metric,

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2). \tag{3.12}$$

This is the metric defined in (3.1), out of convenience we will work in Euclidean space, selecting the Poincaré coordinates. As before we know that the conformal boundary lies at $z = 0$ and as $z \to \infty$ approaches a "horizon" (a Cauchy horizon). Consider a massive scalar field obeying the Klein-Gordon equation but in our curved space,

$$\nabla^2\varphi - m^2\varphi = 0, \tag{3.13}$$

with ∇ being the covariant derivative and the action defined as

$$S = -\frac{1}{2}\int \sqrt{|g|}\left(g^{AB}\partial_A\varphi\partial_B\varphi + m^2\varphi^2\right). \tag{3.14}$$

Note here that our intrinsic metric $\eta_{\mu\nu}$ is just the Minkowski metric, and we have denoted our extrinsic metric to be g_{AB}. Due to our Poincaré isometry in the metric we take the ansatz,

$$\varphi(x^A) = e^{ik_\mu x^\mu} f(z), \tag{3.15}$$

forming a basis for a general solution. Plugging our ansatz into (3.13) we evaluate as follows, starting with the second order covariant derivative

$$\begin{aligned}
\nabla^2\varphi =& g^{AB}\nabla_A(\partial_B\varphi),\\
=& g^{AB}(\partial_A(\partial_B\varphi) - \Gamma^C_{AB}(\partial_C\varphi)),\\
=& g^{\vec{x}\vec{x}}(\partial_{\vec{x}}(\partial_{\vec{x}}\varphi) - \Gamma^z_{\vec{x}\vec{x}}(\partial_z\varphi)) + g^{tt}(\partial_t(\partial_t\varphi) - \Gamma^z_{tt}(\partial_z\varphi)) + g^{zz}(\partial_z(\partial_z\varphi) - \Gamma^z_{zz}(\partial_z\varphi)),\\
=& \frac{z^2}{R^2}\left[-k^2 e^{ik_\mu x^\nu} f(z) + e^{ik_\mu x^\nu} f''(z) - \frac{d-1}{z} e^{ik_\mu x^\nu} f'(z)\right].
\end{aligned} \tag{3.16}$$

Plugging this into (3.13) and dividing through by $\frac{z^2}{R^2}e^{ik_\mu x^\nu}$ we obtain:

$$f''(z) - \frac{d-1}{z} f'(z) - \left(k^2 + \frac{m^2R^2}{z^2}\right) = 0. \tag{3.17}$$

This is solved using Bessel functions such that

$$f(z) = az^{\frac{d}{2}}K_v(kz) + bz^{\frac{d}{2}}I_v(kz), \qquad v = \sqrt{\frac{d^2}{4} + m^2R^2}, \tag{3.18}$$

where a, b are constants. By going close to the boundary at $z = 0$ we have two independent solutions $f(z) = z^{\triangle}$ and $f(z) \sim z^{d-\triangle}$. Expanding the Bessel functions as power series in this limit we get the following linear combination

$$\begin{aligned}
f(z) &= \tilde{a}z^{d-\triangle}(1 + ...) + \tilde{b}z^{\triangle}(1 + ...),\\
\triangle\ (\triangle\ - d) &= m^2R^2, \qquad \triangle > \frac{d}{2}.
\end{aligned} \tag{3.19}$$

Where in the second line we have chosen to constrain $\triangle$ such that only one of the roots of the mass is considered. Note that, as mentioned earlier, we can have slightly negative mass - while this does result in exponentially growing modes this is only at long wavelengths. As we are considering short wavelengths where mass is unimportant we allow this slightly tachyonic set of solutions. We also impose a boundary condition at $z \longrightarrow \infty$ such that $\varphi = 0$, meaning we should set $b = 0$ for sensible dynamics (this condition is essentially saying we have no incoming radiation).

Going back to φ one can write this as

$$\begin{aligned}
\varphi(x, z) =& \varphi_0(x)z^{d-\triangle} + \varphi_2(x)z^{d+2-\triangle} + ...\\
&+ \varphi_d(x)z^{\triangle} + ...
\end{aligned} \tag{3.20}$$

where the first line of (3.20) is from the power series with $\tilde{a}$ coefficient and the second is that with $\tilde{b}$ coefficient. It is worth noting that these φ_i all live on boundary $z = 0$, i.e. the

x-directions are the boundary directions. Because we have a boundary condition fixing $b = 0$ one can find that this actually relates our leading functions such that,

$$\varphi_d(x) \sim \int d^d y \frac{1}{|x-y|^{2\triangle}} \varphi_0(x). \tag{3.21}$$

Now this is actually a familiar sight - (3.21) appears to be the two point function (correlator). Now previously we have used $\triangle$ to refer to the scaling dimension, but here it has been introduced as some power in our expansions.

Consider the rescaling $z \to \lambda z$, and $x^\mu \to \lambda x^\mu$ which remains invariant when we restrict this to the boundary. We want $\varphi(x, z)$ to be invariant under rescaling, implying that when we restrict to the boundary, the scaling of our functions φ_0 and φ_d should adjust accordingly, which we can read from (3.20)

$$\varphi_0 \to \lambda^{d-\triangle} \varphi_0(\lambda x), \quad \varphi_d(x) \to \lambda^{\triangle} \varphi_d(\lambda x). \tag{3.22}$$

Now consider the action

$$S = \frac{1}{2} \int_{\mathcal{M}} d^d x dz \sqrt{|g|} \varphi (\nabla^2 \varphi - m^2 \varphi) - \frac{1}{2} \int_{\partial \mathcal{M}} dS^A \varphi \partial_A \varphi. \tag{3.23}$$

where we've just added a surface term to the action (3.14). Whilst it does not effect the equations of motion to add this surface term, it can create a stationary action. The on-shell action is now just the surface term on the boundary

$$S_{OS} = -\frac{1}{2} \int_{\partial \mathcal{M}} dS^A \varphi \partial_A \varphi = -\frac{1}{2} \lim_{z \to 0} I(z). \tag{3.24}$$

where $I(z)$ is the on-shell action with the (3.20) inserted. This comes out as

$$S_{OS} = -\int d^d x R^{d-1} [(d - \triangle) \varphi_0(x)^2 z^{d-2\triangle} + ... + d\varphi_0(x) \varphi_d(x) z^0 + ...], \tag{3.25}$$

in the $z \to \infty$ limit. Noting from (3.19) that $\triangle > \frac{d}{2}$, we can see that in this limit the leading order terms are divergent. However, we did have more freedom to add additional surface terms to the action which allow us to cancel these divergences,

$$+ \int_{\partial \mathcal{M}} dS (\alpha \varphi^2 + ...) \tag{3.26}$$

where parameters like α can be tuned to make our on-shell action finite. These terms in (3.26) are called *counterterms*. These counterterms are required to make the theory regular by cancelling out the divergent terms from the on-shell action.

This process is renormalisation (we don't go into detail here), but we obtain the renormalised on-shell action

$$\begin{aligned} S_{renorm,OS} &= \frac{2\triangle - d}{2} R^{d-1} \int d^d x \varphi_0(x) \varphi_d(x), \\ &\sim \int d^d x \int d^d y \frac{1}{|x-y|^{2\triangle}} \varphi_0(x) \varphi_0(y). \end{aligned} \tag{3.27}$$

So we are now in a position to talk about how the AdS/CFT correspondence works given this dynamical behaviour. We know that the CFT lives on the conformal boundary of the AdS space containing the bulk gravity theory. Each bulk gravitational field seems to correspond to a scaling generator, for example, a scalar field in the bulk of mass m corresponds to a scalar in the CFT with scaling dimension $\triangle$: $\triangle\ (\triangle\ -d) = R^2m^2$. In other words, the dynamics in the interior are dual to operators on the boundary.

Take the dimension to be $d = 2$. The bulk require boundary conditions - we saw that $\varphi_0(x)$ determined the boundary behaviour entirely. Call this $\varphi_0(x) = J(x)$. In our CFT we can a source for our dual scalar operator: $J(x)$ such that the dimension of this operator is $d-\triangle$.

Take the gravitational path integral is given by Hawking [40]

$$Z_{grav}[J] = \int D[g;\varphi]e^{i\int\sqrt{|g|}\left(\frac{R}{16\pi G}+(\partial\varphi)^2-m^2\varphi^2+\dots\right)}|_{\varphi_0,g_0}, \tag{3.28}$$

where φ_0, g_0 are determined by J. Note that this is the path integral for quantum gravity and not for the semiclassical approximation of gravity. For a CFT we also define the path integral,

$$Z_{CFT}[J] = \int D\Phi e^{i\int(\mathcal{L}_{CFT}+J\Phi)}, \tag{3.29}$$

with the J term denoting sources.

The AdS/CFT correspondence tells us that (3.27) with appropriate boundary conditions is dual to the generating function of correlators for the field theory operator, see work by Witten [48]:

$$Z_{grav}[J] = Z_{CFT}[J]. \tag{3.30}$$

We understand a great deal more in the semiclassical approximation in which $Z_{grav} \sim e^{iS_{reg,OS}[\varphi_0]}|_{\varphi_0=J}$ (we've use regulated and renormalised interchangeably here). The regulatory terms in the gravitational theory relate to the regulatory terms in the CFT, i.e.

$$Z_{CFT}[J] \approx e^{iS_{reg,OS}[\varphi_0=J]}. \tag{3.31}$$

This we know to be the generating function of correlators. To compute the correlators we recall that J is the source for Φ, and so we have to functionally vary (3.1) with respect to J. For the two-point function this is as follows

$$\begin{aligned}\langle\Phi(x)\Phi(y)\rangle &= -\frac{1}{Z_{CFT}}\frac{\partial^2}{\partial J(x)\partial J(y)}Z_{CFT}[J]|_{J=0},\\ &= -\frac{1}{Z_{grav}}\frac{\partial^2}{\partial\varphi_0(x)\partial\varphi_0(y)}Z_{grav}[\varphi_0]|_{\varphi_0},\\ &= \frac{c}{|x-y|^{2\triangle}}.\end{aligned} \tag{3.32}$$

So we see that one can derive the two-point function with the expected result but from a gravitational calculation using this correspondence. In the second line of (3.32) we employ the AdS/CFT correspondence claim (3.30). In the last line of the above we have simply performed the calculation as we know $S_{reg,OS}$ from (3.27) and this is something that we can compute. One of the key results from the correspondence is that the metric is in fact dual to the energy momentum tensor. The constant c can be determined from the gravity side via the condition on g_0, which implies that for the energy-momentum tensor two-point function we have

$$\begin{aligned}\langle T_{\mu\nu}(x)T_{\alpha\beta}(y)\rangle &= \frac{R^{d-1}}{16\pi G}\frac{1}{|x-y|^{2d}}, \\ &= \frac{c}{|x-y|^{2d}}.\end{aligned} \tag{3.33}$$

Note that the scale dimension of the energy-momentum operator is $\triangle = d$ - this allows us to find the form of c, which is a well-known quantity in CFT called the central charge for $d = 2$ but also as the conformal anomaly more generally:

$$c = \frac{R^{d-1}}{16\pi G}. \tag{3.34}$$

3.1.2 Example: Type-IIB String Theory in AdS and $\mathcal{N} = 4$ Super-Yang-Mills

We've defined the AdS/CFT correspondence, delved into some of the dynamics in AdS and considered how the correspondence relates these dynamics to properties of the CFT. Now we consider the one of the most studied examples of the AdS/CFT correspondence as we look to demonstrate the connection between type-IIB string theory compactified on $AdS_5 \times S^5$ with $\mathcal{N} = 4$ super-Yang-Mills theory, following [2].

We begin with type-IIB string theory - in the low energy limit this is a supergravity theory, combining general relativity and supersymmetry, in which our theory contains gravitinos of the same chirality. The gravitino is a massless spinor-vector particle and the gauge (spin $\frac{3}{2}$) fermion supersymmetric counterpart to the graviton. For type IIB in 10-dimensional Minkoski space, take N D3-branes and stack them in parallel within close proximity to one another. Each of these D3-branes is a $(3+1)$-dimensional plane embedded in our $(9+1)$-dimensional flat spacetime. D-branes here are defined in accordance with string perturbation theory in that they are just surfaces that may act as endpoints for open strings.

We have two possible perturbative excitations here - the open string and the closed string. The closed strings are excitations of the vacuum, whereas the open strings ending on our D-branes encode the excitations that take place on these surfaces.

Taking the system at low energy, we only take massless string excitations. For the closed string massless states we may write a low-energy effective Lagrangian describing type-IIB supergravity. For the open string massless states a similar low-energy Lagrangian can be written down but it is the Lagrangian of $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory.

A complete effective action for the massless modes is of the form

$$S = S_{bulk} + S_{brane} + S_{int}, \tag{3.35}$$

where each of the terms are defined as follows:

- S_{bulk}: this is the action of 10-dimensional supergravity (this may come with higher derivative corrections and while only containing massless fields it must account for integrating out massive fields).
- S_{brane}: this is the brane action defined on the $(3+1)$-dimensional brane world-volume, containing the $\mathcal{N} = 4$ super-Yang-Mills Lagrangian in addition to higher derivative corrections.
- S_{int}: this encodes the interactions between brane and bulk modes, in which we obtain the leading term by covariantising the brane action and introducing the background metric for the brane.

One can then expand the bulk action as a free quadratic part describing free propagation of massless modes (including the graviton), plus the interaction terms

$$S_{bulk} \sim \frac{1}{2\kappa^2} \int \sqrt{g}\mathcal{R} \sim \int (\partial h)^2 + \kappa(\partial h)^2 h + ... \tag{3.36}$$

with metric $g = \eta + \kappa h$. Note that explicit dependence on the graviton has been indicated here.

The interacting action S_{int} is also proportional to positive powers of κ, and in taking low energy limit these terms drop out. Intuitively this makes sense as gravity becomes free at long distances. In low energy limit, given that S_{int} vanishes we now have

$$S = S_{bulk} + S_{brane}, \tag{3.37}$$

and because there is no interaction between the two actions now, i.e. between the bulk and the brane we see that we have two decoupled systems in the low energy limit:

- Free gravity in the 10-dimensional bulk theory.
- $mathcalN = 4$ SYM gauge theory on the branes.

This is a key point for outlining the correspondence in this example. Adopting another perspective, one can view the D-branes as massive charged objects that act as a source for various supergravity fields. We can find a D3-brane solution of supergravity in the form

$$\begin{aligned} &ds^2 = f^{-\frac{1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2), \\ &F_5 = (1+*)dtdx_1dx_2dx_3df^{-1}, \\ &f = 1 + \frac{R^4}{r^4}, \qquad R^4 \equiv 4\pi g_s \alpha'^2 N. \end{aligned} \tag{3.38}$$

If we take E_p to denote the energy of an object as measured by an observer at constant r and take E as the energy measured by an observer at infinity, it is possible to relate these by a redshift factor,

$$E = f^{-\frac{1}{4}} E_p. \tag{3.39}$$

This implies that an object brought increasingly closer to $r = 0$ would have increasingly lower energy from the view of the observer at infinity. Now taking the low energy limit in the background solution (3.38), the observer at infinity will witness two kinds of energy excitations - massless particles propagating in bulk region with wavelengths that become very large or we have any form of excitation as we approach $r = 0$.

In the low energy limit we see that these two types of excitation decouple from each other such that the bulk massless particles decouple from the near horizon limit (approaching $r = 0$). This is evident as the wavelength of such particles in this limit will be much greater than the gravitational size of the brane. Similarly, the excitations living near the near horizon limit find it increasingly difficult to climb the gravitational potential and escape to the asymptotic region. This implies that in the low energy theory we can divide the theory into two decoupled pieces: the free bulk supergravity theory and the near horizon region of geometry.

In the near horizon region we can set $r << R$ and thus $f \sim \frac{R^4}{r^4}$, meaning that we recover

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2. \tag{3.40}$$

But this is just the $AdS_5 \times S^5$ geometry.

From both the point of view of the field theory of open strings on the brane and the view of a supergravity description we have two decoupled theories in low energy. In both cases one of the decoupled systems is supergravity in flat space and thus it is only natural to identify the second system appearing in each of the two systems with one another. This leads us to the conjecture: *$\mathcal{N} = 4U(N)$ SYM in (3+1)d is dual to type IIB superstring theory on $AdS_5 x S^5$*. A more detailed description of how these limits are specified can be found from Maldacena [2].

3.2 Entanglement Entropy

Having defined the AdS/CFT correspondence and provided some examples to further elucidate on the nuances of such a duality, we want to explore an area in which this conjecture has been applied to provide new insight into modern physics. One such application heavily involves the computation of entanglement entropy, and so we take a brief interlude from AdS/CFT to provide an overview of some entanglement entropy notions.

3.2.1 Defining Entanglement Entropy

To begin to define entanglement entropy let us define a bipartite quantum system - our total system is divided into two components A and B. We can write the Hilbert space of this system as follows

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{3.41}$$

A typical wavefunction will be of the form

$$\Psi = \sum_n \psi_n(A)\chi_n(B). \tag{3.42}$$

If we can't write the state $|\Psi\rangle$ as a simple product between a state of A and a state of B, we say that the systems A and B are *entangled*. This can be demonstrated in a simple example - consider a 2-spin system in a 4-dimensional Hilbert space with the following wavefunction

$$\begin{aligned} |\psi\rangle &= \frac{1}{2}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \\ &= \frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle) \otimes (|\downarrow\rangle + |\downarrow\rangle). \end{aligned} \tag{3.43}$$

This has clearly been separated into a simple product of a state in A and a state in B. As this is possible we say that (3.43) defines a *pure state* - one that is not entangled. If this were not possible we would see entanglement, something we can actually quantify by this measure we call entanglement entropy. To compute this we take the density matrix of our state

$$\rho = |\Psi\rangle \langle\Psi| \,, \tag{3.44}$$

and find the reduced density matrix of subsystem A by "tracing out" subsystem B:

$$\rho_A = Tr_B \, |\Psi\rangle \langle\Psi| \,. \tag{3.45}$$

Using this we can now define the Von-Neumann entropy (or fine-grained entropy) of A with the rest of the total system as

$$S_A = -Tr_A(\rho_A \log \rho_A). \tag{3.46}$$

This measures the entanglement between subsystems A and B, and as one might expect, if $S_A = 0$ this implies that the reduced density matrix ρ_A describes a pure state and $|\Psi\rangle$ can be written as a simple product. For any pure state $|\Psi\rangle$ we trivially have that $S_A = S_B$. However, if the total system AB is in a mixed state (there is entanglement present) we generally have $S_A \neq S_B$.

Again, we consider an example of this - take the state to be

$$|\psi\rangle = \cos(\theta) \, |\downarrow\uparrow\rangle + \sin(\theta) \, |\uparrow\downarrow\rangle\rangle \,, \tag{3.47}$$

which is clearly entangled. We compute the density matrix

$$\begin{aligned} \rho = |\psi\rangle \langle\psi| = \cos^2(\theta) \, |\downarrow\uparrow\rangle \langle\downarrow\uparrow| + \sin^2(\theta) \, |\uparrow\downarrow\rangle \langle\uparrow\downarrow| \\ + \sin(\theta)\cos(\theta)(|\downarrow\uparrow\rangle \langle\uparrow\downarrow| + |\uparrow\downarrow\rangle \langle\downarrow\uparrow| \,. \end{aligned} \tag{3.48}$$

Now we can compute the reduced density matrix from this

$$\rho_A = Tr_B \, |\psi\rangle \langle\psi| = \cos^2(\theta) \, |\downarrow\rangle \langle\downarrow| + \sin^2(\theta) \, |\uparrow\rangle \langle\downarrow| \,. \tag{3.49}$$

We now have everything we need to compute the entanglement entropy:

$$S_A = -\cos^2(\theta) \log(\cos^2(\theta)) - \sin^2(\theta) \log(\sin^2(\theta)), \tag{3.50}$$

from which one can also show (due to varying θ) that we have different levels of entanglement. Our maximally entangled state would be given by:

$$\theta = \frac{\pi}{4} \implies |\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\,. \tag{3.51}$$

Entanglement entropy in general has two crucial properties:

1. **Subadditivity:**
$$|S_A - S_B| \leq S_{AB} \leq S_A + S_B. \tag{3.52}$$

2. **Strong subadditivity:**
$$\begin{aligned} S_{AC} + S_{BC} &\geq S_{ABC} + S_C, \\ S_{AC} + S_{BC} &\geq S_A + S_B. \end{aligned} \tag{3.53}$$

In (3.52) this is a weaker condition, only considering a bipartite system AB, whereas in (3.53) we have a much stronger property on a tripartite system ABC.

3.2.2 Entanglement Entropy in Quantum Many-Body Systems

Again we consider a quantum system AB, defining the Hamiltonian to be

$$H = H_A + H_B \tag{3.54}$$

from which we deduce that the ground state of this system is unentangled. For any initial state $|\psi(t = 0)\rangle$ that begins unentangled, it will remain this way regardless of any unitary evolution. We add some coupling (interaction) between the two subsystems by modifying the Hamiltonian

$$H = H_A + H_B + H_{AB}, \tag{3.55}$$

giving us a ground state that is entangled. This means that for our unentangled initial state above, we expect this to become more entangled under time evolution.

For typical systems we expect the Hamiltonian (including the interaction Hamiltonian H_{AB}) to be local. Because of this we expect the interactions to only have support in a region of the boundary, say of subsystem A, ∂A.

Considering Figure 3.1, we see that H_{AB} only involved degrees of freedom near ∂A, specifically in some ϵ-region where ϵ denotes the lattice spacing (short distance or IR cutoff). One finds in general that in the ground state of the localised Hamiltonian we have

$$S_A = C\frac{\text{Area}(\partial A)}{\epsilon^{d-2}} + ... \tag{3.56}$$

for a d-dimensional system, where the leading term here is universal and the corrections that follow correspond to long-range entanglement. We know that entanglement between A and $A^c = B$ is hugely dominated by short-range entanglement in the area ∂A where H_{AB} is supported.

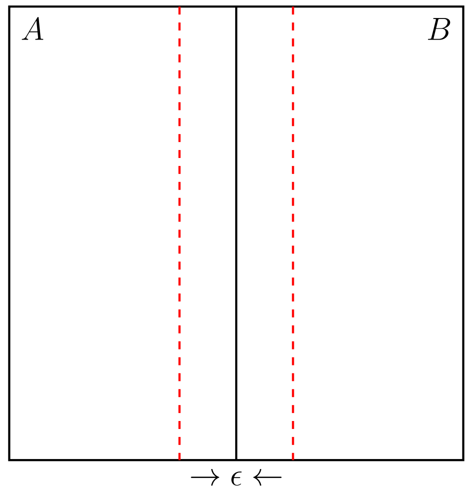

Figure 3.1: Due to locality, the support of H_{AB} lies withing some ϵ-region of the boundary ∂A.

Understanding more about the subleading terms in (3.56) has shown that the contribution from this long-range entanglement can actually characterise a system. The main one that we will consider is the characterisation of degrees of freedom of a QFT.

Consider a $(1+1)$-dimensional CFT with central charge c. For a subsytem A of length l one finds that

$$S_A = \frac{c}{3} \log \left(\frac{l}{\epsilon} \right). \tag{3.57}$$

A very thorough treatment of this is provided by Ryu and Takayanagi in [10] with increased generality.

3.3 The Holographic Derivation of Entanglement Entropy

In 2006, Ryu and Takayanagi [8] published new insight on the AdS/CFT correspondence that provided a holographic derivation of the entanglement entropy in CFTs. The premise of this was the ability to compute the entanglement entropy in a $(d+1)$-dimensional CFT from a D-dimensional minimal surface of a dual AdS_{d+2} space.

We know that for a 1-dimensional quantum many-body system, this is essentially a 2-dimensional CFT if the system is at criticality, a result shown by Ryu and Takayanagi [10] shows that the entanglement entropy is given by,

$$S_A = \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \left(\frac{\pi l}{L} \right) \right), \tag{3.58}$$

which is a well known result for CFTs. To be more specific here, we are describing the entropy for a subsystem A or a full bipartite system AB (which is the 1-dimensional boundary of a circle). Here l is the length of the subsytem A, L is the length of the full system, a is the lattice-spacing or ultra violet (UV) cutoff, and c is the central charge of the CFT.

Away from criticality, we have another known result [10] for the entanglement entropy

$$S_A = \frac{c}{6}\mathcal{A}\log\left(\frac{\xi}{a}\right), \tag{3.59}$$

where $\mathcal{A}$ is the number of boundary points, and ξ is the correlation length. For the setup above we have a CFT on a line with two boundary points so $\mathcal{A} = 2$, and we get that

$$S_A = \frac{c}{3}\log\left(\frac{l}{a}\right). \tag{3.60}$$

The key result of this proposal [8] is the Ryu-Takayanagi (RT) formula. The proposal is as follows; if one defines the entanglement entropy S_A of a CFT, say $\mathbb{R}^{1,d}$, then for the subsystem A with some $(d-1)$-dimensional boundary ∂A we have the "area law":

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}. \tag{3.61}$$

This makes use of a d-dimensional minimal surface in AdS_{d+2} with boundary ∂A, defined by $gamma_A$. The minimal surface acts as a form of holographic screen from the perspective of an observer only accessible to subsystem A. It is noteworthy that (3.61) is reminiscent of the definition of the Bekenstein-Hawking entropy

$$S_{BH} = \frac{\text{Area(horizon)}}{4G_N}, \tag{3.62}$$

originally proposed by Bekenstein in 1973 [15] and then later more formally by Hawking in 1976 [16].

Here our objective is to reproduce the above results for CFTs from the AdS side, indeed giving a holographic derivation of entanglement entropy. We will also prove the strong subadditivity (3.53) requirement for the entanglement entropy defined in this way.

Heuristically, what we gain from this formula are the following conceptual dualities: that spacetime corresponds to entanglement, and that geometry corresponds to quantum information.

3.3.1 Proving Strong Subadditivity

To demonstrate the power and application in this relationship of the RT formula, by considering the minimal surfaces we can greatly simplify the proof that subadditivity holds for entanglement entropy. Strong subadditivity is given by two conditions (3.53), we'll tackle these one at a time in $(1+1)$-dimensions.

Consider three subsytems on a line A, B, and C as depicted in Figure 3.2. We will go about proving subadditivity be making comparisons between the minimal surfaces γ_{AC} and γ_{BC} with γ_1 and γ_2 as shown. We can quite clearly see that

$$\gamma_{AC} + \gamma_{BC} = \gamma_1 + \gamma_2. \tag{3.63}$$

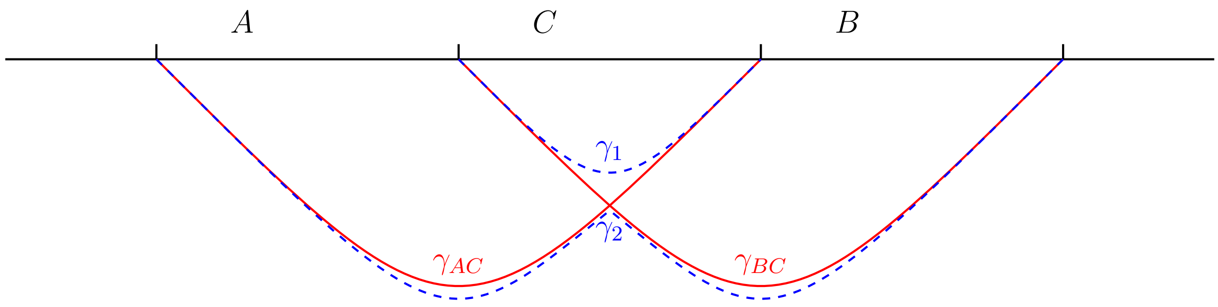

Figure 3.2: The subsystems A, C, and B lie consecutively on the line, with minimal surfaces (thick yellow lines) γ_{AC} and γ_{BC} for AC and BC respectively. We can also define minimal surfaces (dashed blue lines) γ_1 and γ_2 as pictured.

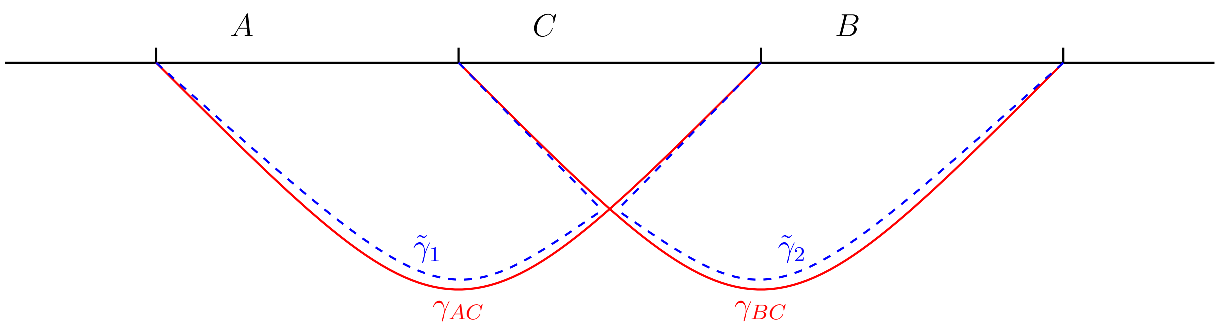

Figure 3.3: This setup is as above except we have redefined the minimal surfaces γ_1 and γ_2.

Then, if we define γ_{ABC} and γ_C such that both of these are minimal surfaces (the index describing the subsystems covered), we can see that the following also hold

$$\begin{aligned} \gamma_1 &\geq \gamma_C, \\ \gamma_2 &\geq \gamma_{ABC}. \end{aligned} \tag{3.64}$$

Putting these back into (3.63), we have

$$\begin{aligned} \gamma_{AC} + \gamma_{BC} = \gamma_1 + \gamma_2 &\geq \gamma_{ABC} + \gamma_C, \\ \implies S_{AC} + S_{BC} &\geq S_{ABC} + S_C, \end{aligned} \tag{3.65}$$

meaning we see that the first statement of subadditivty holds.

Now moving to the setup shown in Figure 3.3 we see that again

$$\gamma_{AC} + \gamma_{BC} = \tilde{\gamma}_1 + \tilde{\gamma}_2. \tag{3.66}$$

Then with γ_A and γ_B being minimal surfaces we see that,

$$\begin{aligned}\tilde{\gamma}_1 &\geq \tilde{\gamma}_A,\\ \tilde{\gamma}_2 &\geq \tilde{\gamma}_B.\end{aligned} \tag{3.67}$$

This can then lead to the second requirement as such

$$\begin{aligned}\gamma_{AC} + \gamma_{BC} = \tilde{\gamma}_1 + \tilde{\gamma}_2 &\geq \gamma_A + \gamma_B,\\ \implies S_{AC} + S_{BC} &\geq S_A + S_B,\end{aligned} \tag{3.68}$$

Without using this idea of holographic entanglement entropy, the full proof is actually complicated and highly non-trivial [41]. It is greatly simplified by this holographic approach using the RT formula [8].

3.3.2 Calculating Entanglement Entropy Holographically

We start with a lower dimensional case, namely when we have a gravitational theory in AdS_3 with radius R that is dual to a $(1+1)$-dimensional CFT with the central charge

$$c = \frac{3R}{2G_N^{(3)}} \tag{3.69}$$

We first calculate the entanglement entropy holographically with AdS Poincaré coordinates before performing the computation in global AdS coordinates.

Poincaré Coordinate Calculation in Three Dimensions

The Poincaré metric in this context is

$$ds^2 = \frac{R^2}{z^2}(dz^2 - dt^2 + dx^2). \tag{3.70}$$

To compute the entanglement entropy holographically using (3.61) we must compute the minimal surface γ_A, which in this case will simply be a line (the "area" element is actually just the length of the line here). Intuitively what we will be doing is foliating the AdS_3 space into time slices, which we will divide into a system AB with two constituent subsystems A and $B = A^c$. As our calculation will take place on a fixed time slice we can merrily set $dt = 0$ such that

$$ds^2 = \frac{R^2}{z^2}(dz^2 + dx^2). \tag{3.71}$$

Now we define with a little more specificity our subsystem A to be on the boundary region of the time slice $z = 0$ with x constrained by $-\frac{l}{2} \leq x \leq \frac{l}{2}$. This means we have boundary points at $x = \pm\frac{l}{2}$, and our minimal surface should connect these through the bulk AdS space.

Take this minimal surface to be $X(\tau) = (x(\tau), z(\tau))$, meaning we take the usual definition of the arclength

$$\lambda(\tau) = \int_{\tau_0}^{\tau} |X'(\tau)| d\tau, \tag{3.72}$$

from which one can find $\lambda = \lambda(\tau)$. Here the arclength is an affine parameter, so we ideally would like to parametrise our coordinates using λ to simplify the geodesic equations. To do so, define the new curve $Y(\lambda) = (x(\lambda), z(\lambda))$ such that $Y(\lambda) = RX(\tau(\lambda))$, which allows us to find:

$$\begin{aligned}
&|\dot{Y}(\lambda)| = |\frac{dY}{d\lambda}| = |\frac{dY}{d\tau}\frac{d\tau}{d\lambda}| = R|\frac{dX(\tau(\lambda))}{d\tau}X'(\tau(\lambda))| = R \\
&\dot{Y}(\lambda) = (\dot{x}(\lambda), \dot{z}(\lambda)) \implies |\dot{Y}(\lambda)| = \sqrt{g_{\mu\nu}\dot{Y}^\mu \dot{Y}^\nu} = \frac{R}{z}\sqrt{\dot{x}^2 + \dot{z}^2}.
\end{aligned} \tag{3.73}$$

This gives us the first equation we will need, the arclength condition:

$$\dot{x}^2 + \dot{z}^2 = z^2. \tag{3.74}$$

Because we selected an affine parameter, we use the non-square root definition of the Lagrangian, which we state,

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{1}{2}R^2 z^{-2}(\dot{x}^2 + \dot{z}^2). \tag{3.75}$$

Noting that there is no x-dependence in the Lagrangian, we know we will get a simplified equation of motion via the Euler-Lagrange equation for x - so we do just that

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0, \tag{3.76}$$

but as there is no x-dependence we find that

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{R^2}{z^2}\dot{x} = C, \tag{3.77}$$

where C is a constant. This gives us a second crucial equation

$$\dot{x}(\lambda) = \alpha z(\lambda)^2, \tag{3.78}$$

with $\alpha = \frac{C}{R^2}$. So we take (3.78) and plug this into (3.74) as follows:

$$\begin{aligned}
\dot{x}^2 + \dot{z}^2 = z^2 &\implies \alpha^2 z^4 + \dot{z}^2 = z^2, \\
&\implies \dot{z}^2 = z^2(1 - \alpha^2 z^2), \\
&\implies \dot{z} = z\sqrt{1 - \alpha^2 z^2}.
\end{aligned} \tag{3.79}$$

This is a solveable ordinary differential equation (ODE) and so rearranging the final line of (3.79) we can obtain

$$\int dz \frac{1}{z\sqrt{1 - \alpha^2 z^2}} = \int d\lambda. \tag{3.80}$$

Making the substitution $y = az$ we find

$$\int dz \frac{1}{z\sqrt{1 - \alpha^2 z^2}} = \int dy \frac{1}{y\sqrt{1 - y^2}}. \tag{3.81}$$

Now we see we can implement the substitution $u = \sqrt{1-y^2}$:

$$\begin{aligned} u = \sqrt{1-y^2} &\implies dy = du\left(\frac{-u}{\sqrt{1-u^2}}\right), \\ \implies \int dy \frac{1}{y\sqrt{1-y^2}} &= \int du \left(\frac{-u}{\sqrt{1-u^2}}\right)\frac{1}{u\sqrt{1-u^2}} = -\int du \frac{1}{1-u^2}. \end{aligned} \tag{3.82}$$

But this final result is a known integral meaning we have

$$-\int du \frac{1}{1-u^2} = -\text{arctanh}(u) = \lambda, \tag{3.83}$$

reinserting the right hand side of (3.80). Using the definition of hyperbolic functions it is possible to find a more convenient form of λ. First note that we can manipulate the minus sign as follows

$$u = \tanh(-\lambda) = -\tanh(\lambda) \implies \tanh(\lambda) = -u. \tag{3.84}$$

Now using the definition of the tanh function,

$$\begin{aligned} &\frac{e^{\lambda} - e^{-\lambda}}{e^{\lambda} + e^{-\lambda}} = -u, \\ \implies &e^{\lambda} - e^{-\lambda} = -u(e^{\lambda} + e^{-\lambda}), \\ \implies &e^{\lambda}(1+u) = e^{-\lambda}(1-u), \\ \implies &e^{2\lambda} = \frac{1-u}{1+u}. \end{aligned} \tag{3.85}$$

This gives us a function of λ in terms of u:

$$\lambda = \frac{1}{2}[\log(1-u) - \log(1+u)], \tag{3.86}$$

which we can further specify by fully restoring λ in terms of the z coordinate:

$$\lambda = \frac{1}{2}[\log(1-\sqrt{1-\alpha^2 z^2}) - \log(1+\sqrt{1-\alpha^2 z^2})], \tag{3.87}$$

This is a good place to stop and consider the boundary conditions for this minimal surface we are computing. On the boundary itself $z = 0$ it is clear from (3.87) that the arclength λ diverges - this is expected by our construction, from which we also can see that $\lambda = 0$ at some $z = z_*$ (this is just the point we began measuring arclength which we take to be z value of the saddle point of the minimal surface).

We are now going to proceed with computing the length of the geodesic from $z = z_*$ to the boundary $z = 0$. To do so, we revisit our fixed time metric (3.71), from which we deduce

$$ds^2 = \frac{R^2(\dot{x}^2 + \dot{z}^2)}{z^2} d\lambda^2. \tag{3.88}$$

Using (3.74), this tells us that $ds^2 = R^2 d\lambda^2$, and so we have can find the minimal surface length:

$$L_{\gamma_A} = 2R\int_0^\infty d\lambda. \tag{3.89}$$

As this integral is divergent we cut off the upper limit by introducing a regulator λ_0. Note the factor of two is because we are taking the integral over half of the minimal surface (due to its symmetry). Following this, we look to transform the regulated version of (3.89) into an integral over z (for which we will make use of a short distance cut off or lattice spacing a).

$$\begin{aligned} L_{\gamma_A} =& 2R\int_0^{\lambda_0} d\lambda = 2R\int_a^{z_0} dz\frac{1}{\dot{z}} = 2R\int_a^{z_0} dz\frac{1}{z\sqrt{1-\alpha^2 z^2}}, \\ =& R\left(\log(1-\sqrt{1-\alpha^2 z_*^2}) - \log(1+\sqrt{1-\alpha^2 z_*^2})\right) \\ &- R\left(\log(1-\sqrt{1-\alpha^2 a^2}) - \log(1+\sqrt{1-\alpha^2 a^2})\right), \\ =& R\log\left(\frac{1-\sqrt{1-\alpha^2a^2}}{1+\sqrt{1-\alpha^2a^2}}\right), \\ =& R\log\left(\frac{2-\alpha^2a^2+2\sqrt{1-\alpha^2a^2}}{\alpha^2a^2}\right), \\ =& R\log\left(\frac{2}{\alpha^2a^2} - 1 + \frac{2}{\alpha^2a^2\sqrt{1-\alpha^2a^2}}\right), \\ =& R\log\left(\frac{l^2}{2a^2} - 1 + \frac{l^2}{2a^2}\sqrt{1-\frac{4a^2}{l^2}}\right). \end{aligned} \tag{3.90}$$

Now we take the limit as $a \to 0$ to obtain

$$L_{\gamma_A} = 2R\log\left(\frac{l}{a}\right). \tag{3.91}$$

So we have successfully computed the area element (length) of the minimal surface in the bulk connecting the boundary points of region A in our fixed time slice. We may now use the Ryu-Takayanagi formula (3.61) to compute the entropy

$$S_A = \frac{R}{2G_N}\log\left(\frac{l}{a}\right), \tag{3.92}$$

but we see that we can substitute the definition of the central charge (3.69) to find

$$S_A = \frac{c}{3}\log\left(\frac{l}{a}\right), \tag{3.93}$$

which exactly matches the result for the 2-dimensional CFT (3.60).

Particularly rigorous readers will have identified that in the final line of (3.90) the substitution $\alpha = \frac{2}{l}$ was used. Let us justify this here - note that from (3.78) and (3.79) we have

$$\dot{x} = \frac{dx}{d\lambda} = \dot{z}\frac{dx}{dz},$$
$$\implies \frac{dx}{dz} = \frac{\alpha z^2}{\dot{z}} = \frac{\alpha z}{\sqrt{1-\alpha^2 z^2}}. \tag{3.94}$$

We then manipulate this via a substitution $y = \alpha z$, giving

$$\alpha\frac{dx}{dy} = \frac{y}{\sqrt{1-y^2}} \implies \alpha \int dx = \int dy \frac{y}{\sqrt{1-y^2}}. \tag{3.95}$$

We note that we are free to choose the sign of α and do so to our convenience below, obtaining

$$x = \frac{1}{\alpha}\sqrt{1-\alpha^2 z^2}, \tag{3.96}$$

by solving (3.95). At $z = 0$ we know that $x = \frac{l}{2}$ by construction, and so we find that

$$\alpha = \frac{2}{l} \tag{3.97}$$

We then see that the minimal surface is in fact the upper half circle with radius $\frac{l}{2}$ centred around the origin, as from (3.96), plugging in (3.97), we see that

$$x^2 + z^2 = \left(\frac{l}{2}\right)^2. \tag{3.98}$$

We could have taken this route to compute the entanglement entropy also, but parametrising such that $x = x(z)$,

$$ds^2 = \frac{R^2}{z^2}(1 + x'(z)^2)dz^2, \tag{3.99}$$

consequently giving the entropy

$$S_A = \frac{1}{4G_N} 2R \int_a^{z_*} \frac{1}{z}\sqrt{1 + x'(z)^2}. \tag{3.100}$$

As we are free to scale, take $z \to \frac{l}{2}z$, now

$$S_A = \frac{R}{2G_N} \int_{\frac{2a}{l}}^{1} dz \frac{1}{z\sqrt{1-z^2}}, \tag{3.101}$$

where we have also used (3.94). Let us compute this integral explicitly,

$$\int_{\frac{2a}{l}}^{1} dz \frac{1}{z\sqrt{1-z^2}} = -\frac{1}{2}\left[\log\left(1-\sqrt{1-\frac{4a^2}{l^2}}\right) - \log\left(1+\sqrt{1-\frac{4a^2}{l^2}}\right)\right],$$

$$= \frac{1}{2}\log\left(\frac{1+\sqrt{1-\frac{4a^2}{l^2}}}{1-\sqrt{1-\frac{4a^2}{l^2}}}\right), \tag{3.102}$$

$$= \frac{1}{2}\log\left(\frac{l^2}{2a^2} - 1 + \frac{l^2}{2a^2}\sqrt{1-\frac{4a^2}{l^2}}\right).$$

and again we take the $a \to 0$ limit to obtain,

$$\int_{\frac{2a}{l}}^{1} dz \frac{1}{z\sqrt{1-z^2}} = \log\left(\frac{l}{a}\right). \tag{3.103}$$

And so we see the expected result by plugging this back into (3.101),

$$S_A = \frac{R}{2G_N}\log\left(\frac{l}{a}\right) = \frac{c}{3}\log\left(\frac{l}{a}\right). \tag{3.104}$$

Global AdS Coordinate Calculation in Three Dimensions

The primary demonstration of the holographic derivation of the entanglement entropy in gravitational theories by Ryu and Takayanagi [8] takes place in AdS_3/CFT_2 (as above) and is computed using global coordinates. The explicit computation is given here, we begin with the global AdS_3 metric

$$ds^2 = R^2(-\cosh^2(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)d\theta^2) \tag{3.105}$$

The boundary $\rho = \infty$ is clearly divergent for this metric, and so we introduce a large limit regulator $\rho \leq \rho_*$. In an identical setup to the Poincar e computation, AB denotes our total system, L is the system length, and a is the lattice spacing of the system (short distance cut off.

It is worth noting that at $\rho = \rho_*$, we have

$$ds^2 = R^2(-\cosh^2(\rho_*)dt^2 + \sinh^2(\rho_*)d\theta^2), \tag{3.106}$$

and so see that the regulator can be related, up to some factor, to the system length and lattice spacing as such

$$e^{\rho_*} \sim \frac{L}{a}. \tag{3.107}$$

At $\rho = \rho_*$, define the subsystem A by the region $0 \leq \theta \leq \frac{2\pi l}{L}$. The minimal surface connects boundary points $\theta = 0$ and $\theta = \frac{2\pi l}{L}$ on a fixed time slice, and so

$$ds^2 = R^2(d\rho^2 + \sinh^2(\rho)d\theta^2). \tag{3.108}$$

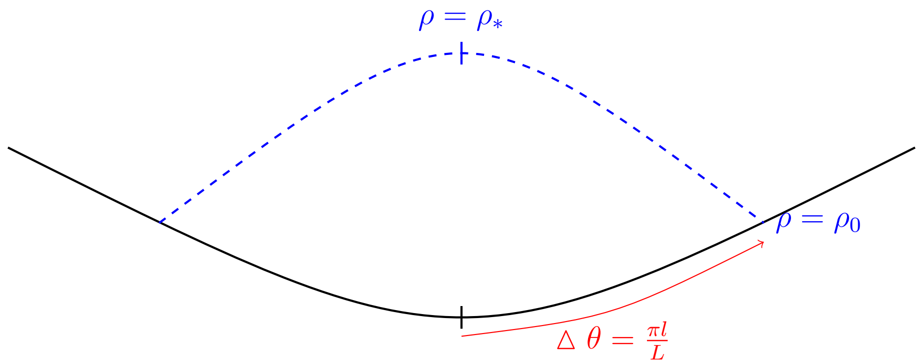

Figure 3.4: The turning point of the minimal surface (blue, dashed line) lies at ρ_*, we want to integrate over angle θ from the midpoint of subsystem A (black, solid line) to the boundary point as shown, as we integrate over half of the desired region we will pick up a factor of 2.

As in the previous calculation, we can affinely parametrise in terms of the arclength λ to obtain the condition

$$\dot{\rho}^2 + \sinh^2(\rho)\dot{\theta}^2 = 1. \tag{3.109}$$

We similarly define the Lagrangian,

$$\mathcal{L} = \frac{1}{2}R^2(\dot{\rho}^2 + \sinh^2(\rho)\dot{\theta}^2), \tag{3.110}$$

and as there is no θ-dependence we use the Euler-Lagrange equations to obtain

$$\dot{\theta} = \frac{\alpha}{\sinh^2(\rho)}. \tag{3.111}$$

Combining (3.109) and (3.111), we find that

$$\dot{\rho} = \sqrt{1 - \frac{\alpha^2}{\sinh^2(\rho)}}. \tag{3.112}$$

The turning point on the minimal surface (minimal geodesic) is given by $\dot{\rho} = 0$. At this point, let us say $\rho = \rho_*$, implying that

$$\alpha = \sinh(\rho_*) \tag{3.113}$$

This can be seen more explicitly in Figure 3.4. We then want to get the explicit form of θ, begin with

$$\frac{d\theta}{d\rho} = \frac{\dot{\theta}}{\dot{\rho}} = \frac{\alpha}{\sinh(\rho)\sqrt{\sinh^2(\rho) - \alpha^2}}. \tag{3.114}$$

This implies that we can compute θ as

$$\theta = \int d\rho \frac{\alpha}{\sinh(\rho)\sqrt{\sinh^2(\rho) - \alpha^2}}. \tag{3.115}$$

Making the substitution $x = -\frac{\alpha \cosh(\rho)}{\sqrt{\sinh^2(\rho) - \alpha^2}}$, we show

$$\begin{aligned}
\frac{dx}{d\rho} &= \frac{d}{d\rho}\left(-\frac{\alpha\cosh(\rho)}{\sqrt{\sinh^2 - \alpha^2}}\right), \\
&= \frac{d}{d\rho}(-\alpha\cosh(\rho)(\sinh^2(\rho) - \alpha^2)^{-\frac{1}{2}}), \\
&= -\alpha\sinh(\rho)(\sinh^2(\rho) - \alpha^2)^{-\frac{1}{2}} + \alpha\cosh^2(\rho)\sinh(\rho)(\sinh^2(\rho) - \alpha^2)^{-\frac{3}{2}}, \\
&= \alpha\sinh(\rho)(\sinh^2(\rho) - \alpha^2)^{-\frac{1}{2}}[-1 + \cosh^2(\rho)(\sinh^2(\rho) - \alpha^2)^{-1}], \\
&= \frac{\alpha\sinh(\rho)}{\sqrt{\sinh^2(\rho) - \alpha^2}}\left(\frac{\alpha^2 + 1}{\sinh^2(\rho) - \alpha^2}\right).
\end{aligned} \tag{3.116}$$

And so, we compute θ as follows using (3.115) and (3.116),

$$\begin{aligned}
\theta &= \int dx \frac{\alpha\sinh(\rho)}{\sqrt{\sinh^2(\rho) - \alpha^2}}\left(\frac{\alpha^2 + 1}{\sinh^2(\rho) - \alpha^2}\right)\frac{\alpha}{\sinh(\rho)\sqrt{\sinh^2(\rho) - \alpha^2}}, \\
&= \int dx \left(\frac{\sinh^2(\rho) - \alpha^2}{(\alpha + 1)\sinh^2(\rho)}\right), \\
&= \int dx \frac{\sinh^2(\rho) - \alpha^2}{(\alpha + 1)(\cosh^2(\rho) - 1)}, \\
&= \int dx \frac{\sinh^2(\rho) - \alpha^2}{\cosh^2(\rho) - 1 + \alpha^2\cosh^2(\rho) - \alpha^2}, \\
&= \int dx \frac{\sinh^2(\rho) - \alpha^2}{\sinh^2(\rho) - \alpha^2 + \alpha^2\cosh^2(\rho)}, \\
&= \int dx \frac{1}{1 + \left(-\frac{\alpha\cosh(\rho)}{\sqrt{\sinh^2(\rho) - \alpha^2}}\right)^2}, \\
&= \int dx \frac{1}{1 + x^2} = \arctan(x) + C.
\end{aligned} \tag{3.117}$$

This gives us the following result for θ

$$\tan(\theta - C) = -\frac{\alpha\cosh(\rho)}{\sqrt{\sinh^2(\rho) - \alpha^2}}. \tag{3.118}$$

The conditions on the minimal surface (Figure 3.4) imply that for $\theta = 0$ we have $\rho = \rho_*$. This implies that

$$\tan(-C) = -\frac{\alpha\cosh(\rho_*)}{\sqrt{\sinh^2(\rho_*)-\alpha^2}} \to \infty, \tag{3.119}$$

telling us we take $-C = \frac{\pi}{2}$. Leaving us with

$$\tan(\theta + \frac{\pi}{2}) = -\frac{\alpha\cosh(\rho)}{\sqrt{\sinh^2(\rho)-\alpha^2}} \implies \cot(\theta) = \frac{\alpha\cosh(\rho)}{\sqrt{\sinh^2(\rho)-\alpha^2}}. \tag{3.120}$$

We also know that we must implement a boundary condition at $\theta = \triangle\,\theta$, $\rho = \rho_0$, so we take

$$\cot(\triangle\,\theta) = \frac{\alpha\cosh(\rho_0)}{\sqrt{\sinh^2(\rho_0)-\alpha^2}}. \tag{3.121}$$

We also know that $\rho_0 \to \infty$, and so we can take $\cosh(\rho_0) \approx \sinh(\rho_0)$,

$$\begin{aligned} &\implies \cot(\triangle\,\theta) \to \alpha = \sinh(\rho_*), \\ &\implies \frac{\cos(\triangle\,\theta)}{\sin(\triangle\,\theta} = \sinh(\rho_*), \\ &\implies \cosh(\rho_*) = \frac{1}{\sin(\triangle\,\theta)}. \end{aligned} \tag{3.122}$$

Now that we have successfully constrained the system using the appropriate boundary conditions, we can compute the geodesic length as an integral over λ,

$$L_{\gamma_A} = \int ds = 2R\int_0^{\triangle\lambda} d\lambda\sqrt{\dot{\rho}^2 + \sinh^2(\rho)\dot{\theta}} = 2R\int_0^{\triangle\lambda} d\lambda. \tag{3.123}$$

One can rewrite (3.112) as such

$$\dot{\rho} = \frac{\sqrt{\sinh^2\rho - \alpha^2}}{\sinh(\rho)}, \tag{3.124}$$

meaning that we can write (3.123) as an integral over ρ using the following

$$d\lambda = \frac{\sinh(\rho)}{\sqrt{\sinh^2\rho-\alpha^2}}d\rho \implies L_{\gamma_A} = 2R\int_{\rho_*}^{\rho_0} d\rho\frac{\sinh(\rho)}{\sqrt{\sinh^2\rho-\alpha^2}}. \tag{3.125}$$

We proceed to make the following substitution $u = \cosh(\rho) \implies du = d\rho\sinh(\rho)$, and so

$$L_{\gamma_A} = 2R\int_{u_*}^{u_0} du\frac{1}{\sinh(\rho)}\frac{\sinh(\rho)}{\sqrt{\cosh^2\rho - 1 - \alpha^2}} = 2R\int_{u_*}^{u_0} du\frac{1}{\sqrt{u^2-\alpha^2-1}}. \tag{3.126}$$

We then substitute $v = \frac{u}{\sqrt{\alpha^2+1}} \implies \frac{dv}{du} = \frac{1}{\sqrt{\alpha^2+1}}$ and obtain,

$$
\begin{aligned}
L_{\gamma_A} &= 2R\int_{v_*}^{v_0} dv \frac{1}{\sqrt{v^2-1}}, \\
&= 2R\left[\operatorname{arccosh}\left(\frac{\cosh(\rho_0)}{\sqrt{\alpha^2+1}}\right) - \operatorname{arccosh}\left(\frac{\cosh(\rho_*)}{\sqrt{\alpha^2+1}}\right)\right], \\
&= 2R\log\left(\frac{\cosh(\rho_0)}{\cosh(\rho_*)} + \sqrt{\frac{\cosh^2(\rho_0)}{\cosh^2(\rho_*)} - 1}\right), \\
&= 2R\log\left(\frac{\cosh(\rho_0) + \sqrt{\cosh^2(\rho_0) - \alpha^2}}{\cosh(\rho_*)}\right), \\
&\approx 2R\log\left(\frac{2\cosh(\rho_0)}{\cosh(\rho_*)}\right) \approx 2R\log\left(\frac{e^{\rho_0}}{\cosh(\rho_*)}\right), \\
&= 2R\log\left(e^{\rho_0}\sin\left(\frac{\pi l}{L}\right)\right).
\end{aligned}
\tag{3.127}
$$

Recalling (3.107), we can take $e^{\rho_0} \sim \frac{L}{a} \sim \frac{L}{\pi a}$, implying

$$
L_{\gamma_A} = 2R\log\left(\frac{L}{\pi a}\sin\left(\frac{\pi l}{L}\right)\right). \tag{3.128}
$$

Now we are ready to compute the entanglement entropy:

$$
S_A = \frac{L_{\gamma_A}}{4G_N^{(3)}} = \frac{R}{2G_N^{(3)}}\log\left(\frac{L}{\pi a}\sin\left(\frac{\pi l}{L}\right)\right), \tag{3.129}
$$

but as we did in the Poincaré case, we can input the central charge and obtain

$$
S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\left(\frac{\pi l}{L}\right)\right), \tag{3.130}
$$

which exactly coincides with the CFT result (3.58).

Poincaré Coordinate Calculation in $d+2$ dimensions

We would like to study this correspondence more generally, and so we are going to compute an example in AdS_{d+2} space which corresponds to a $(d+1)$-dimensional CFT. The example we will compute is the "fixed belt" mentioned in [8] for which Ryu and Takayanagi give a brief outline, with a slightly more explicit calculation in [10], however we will approach this via the method used in the previous two examples.

We first state the Poincaré AdS_{d+2} metric

$$
ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx^2 - dt^2 + \sum_{i=1}^{d-1} dx_i^2\right). \tag{3.131}
$$

We define the straight belt region we will consider as

$$
A_S = \{x^\mu | x \in [-l/2, l/2], x_i \in [-\infty, \infty], \forall i = 1, ..., d-1\} \tag{3.132}
$$

Following a similar procedure to the one outlined in the previous Poincaré coordinate calculation, we are computing the area element of the minimal surface for a fixed time slice and regulated x_i directions at length L,

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx^2 + \sum_{i=1}^{d-1} dx_i^2\right), \quad x_i \in [0, L] \forall i = 1, ..., d-1. \tag{3.133}$$

Now we parametrise in such a way that $z = z(x)$,

$$ds^2 = \frac{R^2}{z^2}\left((1 + z'(x)^2)dx^2 + \sum_{i=1}^{d-1} dx_i^2\right), \tag{3.134}$$

The total area of the disk is then something we can compute, as follows

$$\begin{aligned}\text{Area}(A_S) &= \int \sqrt{g} dx dx_i, \\ &= \left(\prod_{i=1}^{d-1} \int_0^L\right) \int_{-\frac{l}{2}}^{\frac{l}{2}} dx \sqrt{\left[\frac{R^2}{z^2}(1 + z'^2)\right]\left[\frac{R^2}{z^2}\right]^d}, \\ &= R^d L^{d-1} \int_{-\frac{l}{2}}^{\frac{l}{2}} dx \frac{\sqrt{1 + z'^2}}{z^d}.\end{aligned} \tag{3.135}$$

We want to use a Lagrangian and Hamiltonian formalism and so we begin with the action and therefore the Lagrangian via (3.135)

$$S = \int dx \frac{\sqrt{1 + z'^2}}{z^d} \implies \mathcal{L} = \frac{\sqrt{1 + z'^2}}{z^d}. \tag{3.136}$$

Now we define the conjugate momentum P in order to define the Hamiltonian:

$$P = \frac{\partial \mathcal{L}}{\partial z'} = \frac{z'}{z^d \sqrt{1 + z'^2}}. \tag{3.137}$$

Using (3.137) we see that

$$P^2 z^{2d} = \frac{z'^2}{1 + z'^2} \implies 1 - P^2 z^{2d} = \frac{1}{1 + z'^2} \implies \sqrt{1 + z'^2} = \frac{1}{\sqrt{1 - P^2 z^{2d}}} \tag{3.138}$$

We can now use the definition of the Hamiltonian, $H = Pz' - \mathcal{L}$ to find

$$H = \frac{z'^2}{z^d \sqrt{1 + z'^2}} - \frac{1 + z'^2}{z^d} = -\frac{1}{z^d \sqrt{1 + z'^2}} = -\frac{\sqrt{1 - P^2 z^{2d}}}{z^d} \tag{3.139}$$

Noting that H is independent of x, we can take $H = E$ to be constant. We need some further manipulations for the calculation. The first is

$$\frac{\partial H}{\partial P} = -\frac{1}{2z^d} \frac{-2P z^{2d}}{\sqrt{1 - P^2 z^{2d}}} = \frac{P z^d}{\sqrt{1 - P^2 z^{2d}}}. \tag{3.140}$$

And so we can see that from (3.139),

$$\begin{aligned}
&H^2 z^{2d} = 1 - P^2 z^{2d}, \\
\implies &Pz^d = \sqrt{1 - H^2 z^{2d}}, \quad \text{and} \quad Hz^d = \sqrt{1 - P^2 z^{2d}}, \\
\implies &\frac{\partial H}{\partial P} = \frac{\sqrt{1 - H^2 z^{2d}}}{Hz^d} = \frac{\sqrt{\frac{1}{H^2} - z^{2d}}}{z^d}.
\end{aligned} \tag{3.141}$$

As $H = Pz' - \mathcal{L}$ we know that $\frac{\partial H}{\partial P} = z'$, and setting $H = \frac{1}{z_*^d}$,

$$z' = \frac{\sqrt{z_*^{2d} - z^{2d}}}{z^d} \tag{3.142}$$

We would like $z = z_*$ to be a turning point, and therefore we constrain such that

$$\int_0^{\frac{l}{2}} dx = \frac{l}{2} = \int_0^{z_*} dz \left(\frac{1}{z'}\right) = \int_0^{z_*} dz \frac{z^d}{\sqrt{z_*^{2d} - z^{2d}}}. \tag{3.143}$$

The substitution $u = \frac{z}{z_*}$ can be made such that $dz = du z_*$, implies that

$$\begin{aligned}
\frac{l}{2} &= \int_0^1 du z_* \frac{u^d}{\sqrt{1 - u^{2d}}} = z_* \int_0^1 du u^d (1 - u^{2d})^{-\frac{1}{2}}, \\
&= z_* \frac{1}{2d} B\left(\frac{d+1}{2d}, \frac{1}{2}\right) = z_* \frac{1}{2d} \frac{\Gamma\left(\frac{d+1}{2d}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d+1}{2d} + \frac{1}{2}\right)}, \\
&= z_* \frac{1}{2d} \frac{\sqrt{\pi}\Gamma\left(\frac{d+1}{2d}\right)}{\frac{1}{2d}\Gamma\left(\frac{1}{2d}\right)} = \frac{\sqrt{\pi}\Gamma\left(\frac{d+1}{2d}\right)}{\Gamma\left(\frac{1}{2d}\right)} z_*.
\end{aligned} \tag{3.144}$$

Here we have first used the Beta function:

$$\begin{aligned}
&\int_0^1 dx x^{\mu-1} (1 - x^\lambda)^{\nu-1} = \frac{1}{\lambda} B\left(\frac{\mu}{\lambda}, \nu\right), \\
&B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},
\end{aligned} \tag{3.145}$$

and the Gamma function, which obeys the following,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(1+z) = z\Gamma(z). \tag{3.146}$$

We are now ready to compute the definite area of the minimal surface. Go back to the area element

$$\begin{aligned}
\text{Area}(A_S) &= R^d L^{d-1} \int_{-\frac{l}{2}}^{\frac{l}{2}} dx \frac{\sqrt{1 + z'^2}}{z^d}, \\
&= 2R^d L^{d-1} \int_0^{\frac{l}{2}} dx \frac{\sqrt{1 + z'^2}}{z^d}, \\
&= 2R^d L^{d-1} \int_a^{z_*} dz \frac{1}{z'} \frac{\sqrt{1 + z'^2}}{z^d},
\end{aligned} \tag{3.147}$$

where a is the lattice spacing or short distance cut off. Now

$$\left(\frac{1}{z'}\right)\frac{\sqrt{1+z'^2}}{z^d} = \frac{\sqrt{1+\frac{z_*^{2d}-z^{2d}}{z'^{2d}}}}{\sqrt{z_*^{2d}-z^{2d}}}, \tag{3.148}$$

which we can plug back into the area (3.148)

$$\begin{aligned}\text{Area}(A_S) &= 2R^dL^{d-1}z_*^d\int_a^{z_*}dz\frac{1}{z^d\sqrt{z_*^{2d}-z^{2d}}},\\ &= 2R^dL^{d-1}z_*^d\left[\int_0^{z_*}dz\frac{1}{z^d\sqrt{z_*^{2d}-z^{2d}}}-\int_0^{a}dz\frac{1}{z^d\sqrt{z_*^{2d}-z^{2d}}}\right].\end{aligned} \tag{3.149}$$

We compute the two integrals here individually. Again we use the substitution $u=\frac{z}{z_*} \implies dz = duz_*$ to compute the first integral in the final line of (3.149), so

$$\begin{aligned}\int_0^{z_*}dzz^{-d}(z_*^{2d}-z^{2d})^{-\frac{1}{2}} &= \int_0^1 duz_*z^{-d}u^{-d}(1-u^{2d})^{-\frac{1}{2}}z_*^{-d},\\ &= \int_0^1 duu^{-d}(1-u^{2d})^{-\frac{1}{2}}z_*^{1-2d}.\end{aligned} \tag{3.150}$$

This is followed by the substitution $v=\frac{z}{a} \implies dz = adv$, giving us the following result for the second integral in (3.149)

$$\begin{aligned}\int_0^a dz\frac{1}{z^d\sqrt{z_*^{2d}-z^{2d}}} &= \int_0^1 dva^{1-d}v^{-d}(z_*^{2d}-a^{2d}v^{2d})^{-\frac{1}{2}},\\ &\rightarrow \int_0^1 v^{-d}a^{1-d}z_*^{-d},\end{aligned} \tag{3.151}$$

where, in the final line, we take advantage of $a\to 0$. Bringing the constituent parts together, we have

$$\text{Area}(A_S) = 2R^dL^{d-1}z_*^d\left[\int_0^1 duu^{-d}(1-u^{2d})^{-\frac{1}{2}}z_*^{1-2d}-\int_0^1 v^{-d}a^{1-d}z_*^{-d}\right]. \tag{3.152}$$

Let us now compute each of the integrals in (3.152), first of all we have

$$\begin{aligned}\int_0^1 duu^{-d}(1-u^{2d})^{-\frac{1}{2}} &= \frac{1}{2d}B\left(\frac{1-d}{2d},\frac{1}{2}\right),\\ &= \frac{\Gamma\left(\frac{1-d}{2d}\right)\Gamma\left(\frac{1}{2}\right)}{2d\Gamma\left(\frac{1-d}{2d}+\frac{1}{2}\right)},\\ &= \frac{\sqrt{\pi}\Gamma\left(\frac{1-d}{2d}\right)}{2d\Gamma\left(\frac{1}{2d}\right)} = -I.\end{aligned} \tag{3.153}$$

Quite trivially we also see $\int_0^1 dvv^{-d} = -\frac{1}{d-1}$. Let's plug these back into (3.152)

$$
\begin{aligned}
\text{Area}(A_S) &= 2R^d L^{d-1} z_*^{1-d}(-I) + 2R^d L^{d-1} a^{1-d}\left(\frac{1}{d-1}\right), \\
&= \frac{2R^d}{d-1}\left(\frac{L}{a}\right)^{d-1} - 2IR^d\left(\frac{L}{z_*}\right)^{d-1}.
\end{aligned}
\tag{3.154}
$$

Rearranging (3.144) for the form of z_* and using the second identity for the Gamma function in (3.146), we can further manipulate the above, namely

$$
\begin{aligned}
-Iz_*^{1-d} &= \frac{l\Gamma\left(\frac{1-d}{2d}\right)}{2d\Gamma\left(\frac{1}{2d}\right)} \frac{l^{1-d}\Gamma\left(\frac{1}{2d}\right)^{1-d}}{2^{1-d}\pi^{\frac{1-d}{2}}\Gamma\left(\frac{d+1}{2d}\right)^{1-d}}, \\
&= \frac{l^{1-d}2^d\pi^{\frac{d}{2}}}{2d\Gamma\left(\frac{d}{2d}\right)} \frac{\Gamma\left(\frac{1-d}{2d}\right)}{\frac{1-d}{2d}\Gamma\left(\frac{1-d}{2d}\right)\Gamma\left(\frac{d+1}{2d}\right)^{-d}}, \\
&= \frac{2^d\pi^{\frac{d}{2}}\Gamma\left(\frac{d+1}{2d}\right)^d}{(1-d)\Gamma\left(\frac{1}{2d}\right)^d} l^{1-d}, \\
&= \frac{-2^d\pi^{\frac{d}{2}}}{d-1}\left(\frac{\Gamma\left(\frac{d+1}{2d}\right)}{\Gamma\left(\frac{1}{2d}\right)}\right)^2 \frac{1}{l^{d-1}}.
\end{aligned}
\tag{3.155}
$$

From this we can finally compute the entanglement entropy of in $(d+2)$-dimensional AdS using holography and the RT formula in Poincaré coordinates:

$$
S_A = \frac{1}{4G_N^{(d+2)}}\left[\frac{2R^d}{d-1}\left(\frac{L}{a}\right)^{d-1} - \frac{-2^d\pi^{\frac{d}{2}}}{d-1}\left(\frac{\Gamma\left(\frac{d+1}{2d}\right)}{\Gamma\left(\frac{1}{2d}\right)}\right)^2\left(\frac{L}{l}\right)^{d-1}\right] \tag{3.156}
$$

A similar result [8], [10] can be found for the disk $A_D = \{x_i | r \leq l\}$.

Yang-Mills at Finite Temperature

In [8], Ryu and Takayanagi show (as have others prior to this paper) that $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory on $\mathbb{R}$ at finite temperature T is dual to the black hole defined by [23] as

$$
ds^2 = R^2\left[\frac{du^2}{hu^2} + u^2(-hdt^2 + dx_1^2 + dx_2^2 + dx_3^2) + d\Omega_5^2\right], \tag{3.157}
$$

with $h = 1 - \frac{u_0^4}{u^4}$ and $u_0 = \pi T$. We'll briefly provide some commentary on this example purely because it is one of the most fundamental aspects of the AdS/CFT correspondence. There is also a more detailed discussion on black holes in the following chapter for more context around this topic.

We take the same example as the calculation prior to this and look to compute the entanglement entropy for A_S but in SYM at finite temperature. Taking the cut off to be $u \sim z_{-1} \sim a$ one can find the area to be

$$\text{Area}(A_S) = 2R^3L^2 \int_{u_*}^{a^{-1}} du \frac{u^6}{\sqrt{(u^4 - u_0^4)(u^6 - u_*^6)}}, \tag{3.158}$$

with u_* is defined such that:

$$\frac{l}{2} = \int_{u_*}^{\infty} du \left[(u^4 - u_0^4) \left(\frac{u^6}{u_*^6} - 1 \right) \right]. \tag{3.159}$$

We can see that we will get some UV divergences here but we can find the finite part of the entropy in the large l limit regardless:

$$S_{finite} \approx \frac{\pi^2}{2} N^2 T^3 L^2 l = \frac{\pi^2 N^2 T^3}{2} \times \text{Area}(A_S). \tag{3.160}$$

This entropy can be regarded as part of the Hawking-Bekenstein entropy of black 5-branes, proportional to the area of the horizon at $u = u_0$. Because of this we can interpret S_{finite} as thermal entropy contribution to the total entanglement entropy at finite temperature.

In the gravitational description, we can paint more of a picture to visualise how the entropy relates to the black hole. The thermality arises because the minimal surface wraps a part of the black hole horizon. If we expand the size of A until it coincides with the total system, γ_A wraps the horizon completely and $S_A = S_{BH}$ as expected.

The AdS black hole can be dual to an entanglement of two different CFTs at the two boundaries, per Maldacena [24]. It is interesting to look at minimal surfaces connecting them and in the appropriate limit, one can consider the black hole entropy as the entanglement entropy of the CFTs as the minimal surface wraps the horizon.

Chapter 4

Black Holes and AdS/CFT

The consequences of this incredible duality span a large, yet inevitably incomplete, set of research areas in physics - either unlocking novel mechanisms and perspectives within existing problems or bringing about new and exciting ideas for study. For the purposes of our discussion in this report, we will focus on the application of this gauge-gravity duality - the AdS/CFT correspondence - to black holes, beginning with a brief overview of black hole solutions in Minkowski and AdS space, reviewing black hole thermodynamics and entropy, and computing the holographic entanglement entropy via the AdS/CFT correspondence.

4.1 Schwarzschild Solution

Black holes are a prediction of Einstein's theory of general relativity - unavoidable singularities hidden behind a horizon, within which no physical particles can escape due to the very nature of this regions geometry.

The first black hole solution to define would be the Schwarzschild solution - this is the simplest black hole solution we will discuss here and it is incredibly apparent where the curvature singularity lies. The metric for this coordinate system is defined as follows

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{4.1}$$

where we are in (3+1)-dimensions and $d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2$. This is the unique spherically symmetric vacuum solution according to general relativity.

One can also see that clearly there is some singular behaviour present in (4.1), namely that we encounter apparent singularities in the metric coefficients when either $r = 0$ or $r = 2GM$. Now we must recall that these coefficients do depend on the coordinates used to define the metric, and so we must check for one of two possible types of singular behaviour here.

- Coordinate Singularity: this is the case in which we don't actually observe singular behaviour, but more we have used an ill-defined coordinate system for the space. These can be removed simply by transforming to another coordinate system that is well-defined at the point of the coordinate singularity.

- Curvature Singularity: this is more subtle and in the case of a curvature singularity we must check if the geometry itself is out of control. To do so we must check curvature scalars - measures of curvature independent of the coordinate system used - such as the Ricci scalar R or other higher order scalars. If any of these are divergent as we approach the singular point in question we have a true singularity of the spacetime.

For the case at hand, one can check that the point $r = 0$ is the true curvature singularity that defines the black hole, and that $r_s = 2GM$ is a coordinate singularity that defines the location of the black hole's event horizon. We assume here that the reader has the background knowledge regarding Schwarzschild black holes and their maximal extension, including Schwarzschild geodesics, the Penrose diagram and the formation of the black hole from star (photon shell) collapse. Rigourous description of this can be found from Susskind and Lindesay [18] and Carroll [4].

It is important here to state *Birkhoff's theorem* but a full treatment of this can be found by Carroll [4]. The theorem states that for a spherically symmetric, asymptotically flat vacuum, the unique solution must be the Schwarzschild metric. We will later discuss the validity of this uniqueness theorem when we consider the AdS-Schwarzschild solution.

4.2 Black Hole Thermodynamics

During the 1970's it was found that black holes behaved as thermal objects leading largely from Bekenstein and Hawking, meaning they appear to obey parallels of thermodynamic laws - allowing one to define an entropy and a temperature that is indicative of Hawking radiation. This was pivotal in the conceptualisation of the *Hawking information paradox.*

The typical story one begins with for black hole thermodynamics is that of dropping an object into a black hole - we expect a dynamical response to this action as we do so, a rippling at the event horizon that then settles back into a new equilibrium at an increased radius. The rules governing these changes in the black hole geometry due to such actions is somewhat akin to the laws of thermodynamics. Almheiri, Hartman, Maldacena, Shaghoulian, and Tajdini provide a brief summary of black hole thermodynamics in [9], but more detailed outlines can be found by Carroll [4] and Wald [19]. The equation for the response of a black hole to an infalling object is given by [9] as

$$\frac{\kappa}{8\pi G_N} dA = dM - \Omega dJ, \tag{4.2}$$

in which κ is the surface gravity, M is the mass of the black hole, J is the angular momentum, Ω is the rotational velocity of the horizon, A is the area of the event horizon and G_N is Newton's gravitational constant. If one was to postulate that the black hole in fact possessed a temperature $T \propto \kappa$ and an entropy $S \propto A$, then (4.5) becomes identical to the first law of thermodynamics:

$$TdS = dM - \Omega dJ. \tag{4.3}$$

But also in the classical theory, the area of the horizon is ever-increasing for such actions, connecting us to the second law of thermodynamics. In [16], Hawking showed that in the case of general relativity coupled to QFT black holes have a temperature

$$T = \frac{\hbar\kappa}{2\pi}, \tag{4.4}$$

where we are working in units such that $c = k_B = 1$. By noting that the total black hole entropy must have a contribution from the exterior, we can also define the generalised black hole entropy

$$S_{gen} = \frac{A}{4\hbar G_N} + S_{exterior}, \tag{4.5}$$

with $S_{exterior}$ being the entropy of matter and gravitons outside of the black hole (in the semiclassical theory). This also includes a vacuum contribution from the quantum fields themselves. Even with the addition of this quantum term, the generalised entropy we've defined here (4.8) obeys the second law of thermodynamics, in that

$$\triangle\, S_{gen} \geq 0. \tag{4.6}$$

4.2.1 Thermality and Euclidean Black Holes

Recall the Schwarzschild metric (4.1), for a Schwarzschild black hole in $(3 + 1)$-dimensions we know this can be written in terms of the Schwarzschild radius r_s:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{4.7}$$

where $r_s = 2GM$ denotes the Schwarzschild radius. For the moment we can neglect the angular coordinates here, they are not particularly relevant to the following discussion.

We impose the following coordinate transformation

$$r \to r_s \left(1 + \frac{\rho^2}{4r_s^2}\right), \quad t \to 2r_s\tau, \tag{4.8}$$

and expand this for $\rho \ll r_s$. This gives us the "near-horizon" metric:

$$ds^2 \approx -\rho^2 d\tau^2 + d\rho^2. \tag{4.9}$$

Now interestingly, if one has little background understanding of the black hole horizon, one would expect something interesting to happen with the geometry near the horizon. However, we see that if we were to transform coordinates again, say taking $x^0 = \rho \sinh(\tau)$ and $x^1 = \rho cosh(\tau)$, we actually get flat Minkowski space. In other words, to an inwardy free-falling observer Polchinski [20] summarises the perspective with the no drama postulate: "a freely falling observer experiences nothing out of the ordinary when crossing the horizon" - we understand this to be a manifestation of Einstein's equivalence principle that in localised patches of spacetime one cannot detect the existence of a gravitational field. Locally, we think of the horizon as being smooth.

In our near-horizon coordinates, one can show that for an observer at fixed ρ, the trajectory they must follow is that of a uniformly accelerated observer, and hence we know that they will detect thermal radiation. This is known as the Unruh effect [21], describing that for a uniformly accelerated observer (a Rindler observer), they will experience a thermal bath of particles (vacuum fluctuations) at the Unruh temperature.

One way to compute the temperature is via a Wick rotation, so implement the coordinate transformation

$$\begin{aligned} \tau \to i\theta, \qquad x^0 \to i x^0_E, \\ \implies x^0_E = \rho \sin(\theta), \qquad x^1 = \rho\cos(\theta), \end{aligned} \tag{4.10}$$

which moves us to the Euclidean plane, $\mathbb{R}^2$, coordinates. This tells us that an observer at some fixed ρ moves in a circle with length $2\pi\rho$ - this Euclidean time evolution on the circle is the relation that will allow us to compute thermodynamical quantities. For a thermodynamic system, we know that the partition function for a given temperature $T = \frac{1}{\beta}$ is given by

$$Z = Tr[e^{-\beta H}] \tag{4.11}$$

with β denoting the length of Euclidean time evolution. Then, the temperature felt by an accelrated observer will be

$$T_{proper} = \frac{1}{2\pi\rho} = \frac{\hbar a}{2\pi k_B c}, \tag{4.12}$$

where we restore fundamental constants for the sake of the reader's intuition, and a is the proper acceleration of the observer. T_{proper} is the proper temperature felt by the near-horizon observer such that $T_{proper} \to \infty$ at $\rho = 0$ and decreases as $\rho \to \infty$. This decrease in temperature is consistent with what we might expect for thermal equilibrium in the presence of a gravitational potential.

We quote the Tolman relation [22] as we are in a spherically symmetric configuration:

$$T_{proper}\sqrt{-g_{\tau\tau}} = const. \tag{4.13}$$

Therefore, by (4.9), we see that

$$T_{proper}\sqrt{-g_{\tau\tau}} = \frac{\hbar a}{2\pi k_B c}\rho = \frac{\hbar}{2\pi k_B c} \sim \frac{1}{2\pi}. \tag{4.14}$$

If one moves back to (t, r) coordinates (4.7), it is simple to show that this becomes the *Hawking temperature*

$$T_{Hawking} = T_{proper}(r \gg r_s) = \frac{1}{4\pi r_s}. \tag{4.15}$$

This is due to the fact that at large r, $g_{tt} = -1$. Observers far from the black hole will measure this temperature.

Now we want to explore this link between the wick rotated or Euclidean time $t_E : \quad t \to it_E$ - we can see that this has a periodicity that is directly related to the temperature. For a thermal state, we have the partition function

$$Z = Tr[e^{-\beta H}]. \tag{4.16}$$

By the cyclic properties of the trace we can show that any observable, say $Tr[O(t)O(0)e^{-\beta H}]$, is periodic with $t \to t + i\beta$ given we know that $O(t) = e^{iHt}Oe^{-iHt}$. If one returns to Figure 2.4, one can gain some intuition more visually - the time coordinate appears to wrap into a circle under the Wick rotation, which of course will lead to some periodic nature in the temporal coordinate.

We can think more generally about this considering the path integral. A Lorentzian path integral describes real time evolution and a Euclidean path integral describes Euclidean time evolution. The Euclidean geometry evolves for a time β with the trace being interpretted as the identification of both "ends" - the boundary conditions if you will.

Now if we have a strip with length β obeying periodic boundary conditions this is equivalent to a path integral over a cylinder, thus for QFT we calculate Z (4.16) using such a path integral on the Euclidean cylinder on which the angle is given by $\theta = \theta + \beta$.

One way to compute this directly is simply by coordinate changes and demanding a smooth geometry. Consider some d-dimensional black hole metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{4.17}$$

where $d\Omega$ contains our angular coordinates in $d-2$ dimensions. If we Wick rotate $t \to it_E$, we obtain,

$$ds_E^2 = f(r)dt_E^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \tag{4.18}$$

Take the horizon to be at $r = r_0$ and go to the near horizon limit $r = r_0 + \epsilon, \quad \epsilon \ll 1$. In this limit we know that by expansion around ϵ, we have that $f(r) \approx f'(r)|_{r=r_0}\epsilon$ to find the metric:

$$ds_E^2 \approx \epsilon f'(r_0)dt_E^2 + \frac{d\epsilon^2}{\epsilon f'(r_0)}, \tag{4.19}$$

where we have dropped the angular terms as they aren't important for this point. We can obtain the usual polar coordinate metric by taking the coordinate transformation,

$$R = 2\sqrt{\frac{\epsilon}{f'(r_0)}}, \quad \varphi = \frac{t_E f'(r_0)}{2}, \tag{4.20}$$

leading to

$$ds_E^2 \approx dR^2 + R^2 d\varphi^2. \tag{4.21}$$

However, this only defines a smooth geometry in the polar coordinates if $\varphi \to \varphi + 2\pi$ - we require 2π periodicity. If this is the case, then from (4.20) we must have that $t_E \to t_E + \frac{4\pi}{f'(r_0)}$. But we know that if for inverse temperature β as defined by (4.16), we have

$$\beta = \frac{4\pi}{f'(r_0)}, \tag{4.22}$$

which defines our black hole temperature:

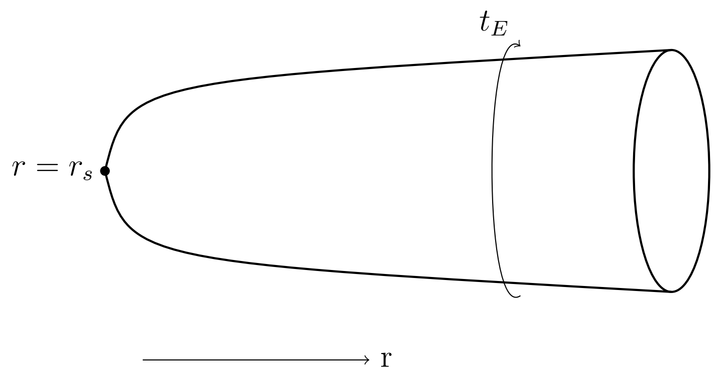

Figure 4.1: The Euclidean Schwarzschild black hole is often referred to as the "cigar", made smooth by adding in the point $r = r_s$.

$$T_{BH} = \frac{1}{\beta} = \frac{f'(r_0)}{4\pi}. \tag{4.23}$$

For a black hole spacetime we can consider a Euclidean Schwarzschild black hole with $t \to it_E$:

$$ds_E^2 = \left(1 - \frac{r_s}{r}\right) dt_E^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{4.24}$$

with $t_E = t_E + \beta$. Taking $R = r - r_s$ to be the radial coordinate in the Euclidean spacetime, the origin must be at $R = 0$ with the radial coordinate in (4.24) being restricted in such a way that $r > r_s$. The Euclidean black hole has no interior in this description.

The setup is shown in Figure 4.1. Following the above procedure for finding β we want to define

$$f(r) = 1 - \frac{r_s}{r} \implies f'(r) = \frac{r_s}{r^2} \tag{4.25}$$

which means, taking our horizon to be the Schwarzschild radius $r = r_s$ we have $f'(r_s) = \frac{1}{r_s}$. Plugging this into (4.22) we get

$$\beta = 4\pi r_s, \tag{4.26}$$

corresponding exactly with (4.18) when taking $T = \frac{1}{\beta}$. This is seen as the temperature to observers who are far from the Euclidean black hole, and we clearly see from Figure 4.1 that this is a circle of circumference β and hence periodic. On this geometry we can think of the path integral as the partition function we know and love from statistical mechanics,

$$Z(\beta) = \text{Euclidean BH Path Integral} \sim e^{-I_{classical}} Z_{quantum}. \tag{4.27}$$

The Einstein action is I in (4.27) and describes the gravity contribution. The quantum contribution is defined for partition functions of quantum fields within this geometry. We also state the generalised entropy using typical thermodynamic formulae

$$S = (1 - \beta\partial_\beta)\log Z(\beta). \tag{4.28}$$

To conclude our discussion on black hole thermodynamics, we consider how a black hole can very much be treated in the same way we would treat an ordinary quantum system, for which thermodynamics has really been the catalyst. What we notice is that the black hole can be (somewhat in an ordinary fashion) be described by a large (but finite) number of degrees of freedom obeying the usual laws of thermodynamics (in addition to physics more generally). As stated by Almheiri et al [9], from the outside a black hole can simlpy be described using Area/$4G_N$ degrees of freedom evolving in a unitary fashion under time evolution. This concept we have already seen run parallel to the AdS/CFT correspondence in the previous chapter.

4.2.2 AdS Black Holes and the AdS/CFT Correspondence

So now we have discussed some tools for manipulations regarding black holes in flat spacetime, let us extend our efforts to AdS. The applications of the AdS/CFT correspondence have progressed many fields but black hole research is certainly at the forefront of that. We start with the 5-dimensional AdS-Schwarzschild metric [28] used by Balasubramanian and Ross

$$ds^2 = -\left(1 + \frac{r^2}{l^2} - \frac{r_0^2}{r^2}\right)dt^2 + \left(1 + \frac{r^2}{l^2} - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \cos^2(\theta)d\Omega_2^2). \tag{4.29}$$

As is now seemingly customary for these calculations, we Wick rotate the temporal coordinate $t \to it_E$, giving

$$ds_E^2 = \left(1 + \frac{r^2}{l^2} - \frac{r_0^2}{r^2}\right)dt_E^2 + \left(1 + \frac{r^2}{l^2} - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \cos^2(\theta)d\Omega_2^2). \tag{4.30}$$

Define $f(r) = 1 + \frac{r^2}{l^2} - \frac{r_0^2}{r^2}$ and note that the roots of this give us coordinate singularities (horizons) in the spacetime. One can find these roots to be

$$\begin{aligned} f(r) = 0 &\implies 1 + \frac{r^2}{l^2} - \frac{r_0^2}{r^2} = 0 \implies (r^2)^2 + l^2r^2 - l^2r_0^2 = 0, \\ &\implies r_\pm^2 = \frac{l^2}{2}\left[-1 \pm \sqrt{1 + \frac{4r_0^2}{l^2}}\right]. \end{aligned} \tag{4.31}$$

We take the r_+ solution (this describes a large black hole, we don't take the r_- small black hole solution as this is unstable), but we want to avoid the conical defect at $r = r_+$. We first want to compute r_0^2:

$$r_0^2 = \frac{l^2}{4}\left[\left(\frac{2r_+^2 + l^2}{l^2}\right)^2 - 1\right] = \left(\frac{2r_+^2 + l^2}{2l}\right)^2 - \frac{l^2}{4} = \frac{r_+^4 + l^2r_+^2}{l^2}. \tag{4.32}$$

Then by the description used to obtain (4.26), we compute the derivative of f taken $r = r_+$,

$$
\begin{aligned}
f'(r_+) =& \frac{2r_+}{l^2} + \frac{2r_0^2}{r_+^3}, \\
=& \frac{2r_+}{l^2} + \frac{2r_+^4 + 2l^2 r_+^2}{r_+^3 l^2}, \\
=& \frac{2(2r_+^2 + l^2}{l^2 r_+}.
\end{aligned}
\tag{4.33}
$$

We again use (4.26) and find that

$$
\beta = \frac{4\pi}{f'(r_+)} = \frac{2\pi l^2 r_+}{2r_+^2 + l^2}. \tag{4.34}
$$

From the final output of the above (4.34), we can see that we need $\frac{2(2r_+^2+l^2)}{l^2 r_+} t_E$ to be 2π-periodic, and with $t_E = t_E + \beta$ we have that the temperature of the AdS-Schwarzschild black hole:

$$
T = \beta^{-1} = \frac{2r_+^2 + l^2}{2\pi l^2 r_+} \tag{4.35}
$$

There is also a final solution called Euclidean thermal AdS, with metric

$$
ds^2 = \left(1 + \frac{r^2}{l^2}\right) dt_E^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2, \tag{4.36}
$$

again with the identification that $t_E \sim t_E + \beta$. For AdS-Schwarzschild we might want to understand these three solutions and in what scenarios they would arise. Following Hartman [29], we can compute the free energy via the on-shell action, this is given by $F = -\frac{1}{\beta}\log(Z)$. We know that for the Einstein action we must consider non-vanishing boundary terms, namely we will need to add in the Gibbons-Hawking-York boundary term and a boundary term to counter divergences (counterterms). The full action Hartman presents is

$$
\begin{aligned}
I_E =& -\frac{1}{16\pi G_N} \int d^5x \sqrt{g} \left(R + \frac{12}{l^2}\right) + \frac{1}{8\pi G_N} \int_{r=\frac{1}{\epsilon}} d^4x \sqrt{\gamma} K + \int_{r=\frac{1}{\epsilon}} d^4x \sqrt{\gamma} L_{ct}[\gamma], \\
&= I_{bulk} + I_{GHY} + I_{ct},
\end{aligned}
\tag{4.37}
$$

noting that here we have used $\gamma_{\mu\nu}$ as the boundary metric and a spacetime cut off $r = \frac{1}{\epsilon}$ has been used where we will take the limit as $\epsilon \to 0$. Define the Einstein equation

$$
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4.38}
$$

where we have included the right hand side of the field equations - κ denoting the Einstein gravitational constant and $T_{\mu\nu}$ being the usual energy-momentum tensor. For the bulk action we are in empty spacetime and so the right hand side of (4.38) vanishes under the trace; we will trace the Einstein field equations as we wish to determine the Ricci scalar R. From reading the bulk action in (4.37) we also can read off the value of the cosmological constant as $\Lambda = -\frac{6}{l^2}$. So taking the trace of (4.38) and plugging in the cosmological constant we obtain $R = -\frac{20}{l^2}$. With this we compute the bulk action

$$\begin{aligned}
I_{bulk} &= -\frac{1}{16\pi G_N}\int_{r<\frac{1}{\epsilon}} d^5x\sqrt{g}\left(R + \frac{12}{l^2}\right), \\
&= \frac{1}{2\pi G_N l^2}\int_{r<\frac{1}{\epsilon}} d^5x\sqrt{g}, \\
&= \frac{1}{2\pi G_N l^2} \in_0^\beta dt_E \int_{r_+}^{\frac{1}{\epsilon}} dr r^3 \int d\Omega_3, \\
&= \frac{\pi\beta}{4G_N l^2}\left[\frac{1}{\epsilon^4} - r_+^4\right].
\end{aligned} \tag{4.39}$$

We plug in the usual definition of the Gibbons-Hawking-York boundary term to find

$$I_{GHY} = -\frac{\pi\beta}{G_N l^2}\left[\frac{1}{\epsilon^4} + \frac{3l^2}{4\epsilon^2} - \frac{r_0^2 l^2}{2}\right]. \tag{4.40}$$

Taking $I_{bulk} + I_{GHY}$ we can consider the divergent terms in this sum to find the required boundary counterterms for finite I_E, which one can show to be

$$I_{ct} = \frac{3}{8\pi G_N l}\int_{r=\frac{1}{\epsilon}} d^4x\sqrt{\gamma}\left(1 + \frac{l^2}{12}R[\gamma]\right), \tag{4.41}$$

with $R[\gamma]$ being the Ricci scalar for the boundary metric - this is also an important distinguishment as we cannot evaluate the energy-momentum tensor to vanish under the trace here.

Overall, we have the Einstein action (finite) given by

$$I_E^{\text{(large black hole)}} = \frac{\pi^2\beta}{8G_N l^2}\left[r_+^2 l^2 - r_+^4 + \frac{3l^4}{4}\right]. \tag{4.42}$$

For the Einstein action we know there is a semiclassical approximation for gravity for the path integral (partition function) in which we include each possible solution, to outline this we could approximately say,

$$Z_{grav} \approx e^{-I_E^{\text{(large black hole)}}} + e^{-I_E^{\text{(small black hole)}}} + e^{-I_E^{\text{(thermal AdS)}}}. \tag{4.43}$$

So we also need I_E for the small black hole solution and the thermal AdS solution. The small black hole is very simple in that we just choose the $r = r_-$ root in (4.32). For the thermal AdS solution, the metric obeys boundary conditions such that as $r \to \infty$ we get

$$ds^2 \to \frac{r^2}{l^2}dt_E^2 + \frac{l^2}{r^2}dr^2 + r^2 d\Omega_3^2, \tag{4.44}$$

Following a similar methodology to the large black hole, one can find for thermal AdS that we get

$$I_E^{(\text{thermal AdS})} = \frac{\pi^2\beta}{8G_N l^2}\left(\frac{3l^4}{4}\right). \tag{4.45}$$

So when we consider transitions between these three solutions, we can immediately rule out the small black hole, as Hartman states [29] it never dominates the canonical ensemble - the free energy of the large black hole by definition will always be smaller, it is a more stable solution and one might expect the small black hole to quickly decay to the large black hole solution, should it arise. We then must consider the large black hole solution and the thermal AdS solution. The partition function takes the form shown in (4.43). As the terms are negative exponents we can see that they blow up (in a finite way) when we substitute in the actions (4.42) and (4.45), and that the bigger of the two actions will dominate the partition function:

$$\log(Z_{grav}) \approx \max\left(I_E^{(\text{large black hole})}, I_E^{(\text{thermal AdS})}\right). \tag{4.46}$$

Because of this we can easily see there must be a transition - the *Hawking-Page transition* - when the two actions are equal for a given β, which reaches criticality at

$$\beta_* = \frac{2\pi l}{3}, \tag{4.47}$$

which occurs at $r_+ = l$. As temperature goes from low to high we go from thermal AdS to the large black hole solution. This implies that in a semiclassical approximation we go from a small number of states when the system has a low temperature, to an abrupt and incredibly large increase in states at high temperature as the criticality threshold for the Hawking-Page transition is crossed. The entropy for both solutions can be computed using

$$S = (1 - \beta\partial_\beta)\log(Z), \tag{4.48}$$

and it is a simple calculation to show that

$$\begin{aligned} S^{(\text{large black hole})} &= \frac{\pi^2 r_+^2}{2G_N}, \\ S^{(\text{thermal AdS})} &= 0. \end{aligned} \tag{4.49}$$

These results are semiclassical, so in reality the entropy for the thermal AdS solution would not be zero, but this emphasises that it is far smaller than the entropy of the large black hole solution and there is a very sharp transition between the two states.

AdS/CFT for a Black Hole at Finite Temperature

Now that we understand a bit more about the nature of a black hole in AdS spacetime - in particular a Euclidean AdS-Schwarzschild black hole, let us demonstrate an example of a holographic entanglement entropy calculation for 3-dimensional AdS. Our system has some temperature given by $T = \frac{1}{\beta}$ and L is the total length of the system such that $\frac{\beta}{L} \ll 1$. At high temperatures it was shown that the gravity dual of the 2-dimensional CFT is the Euclidean BTZ black hole [30]

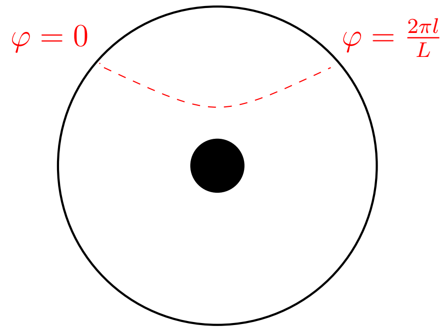

Figure 4.2: The BTZ black hole exists within a foliation of our spacetime bulk where the minimal surface also lies, we define our subsystem A to be a patch of the boundary for this fixed time slice. The outer circle is our boundary $r = \infty$

$$ds^2 = (r^2 - r_s^2)d\tau^2 + \frac{R^2}{r^2 - r_s^2}dr^2 + r^2 d\varphi^2. \tag{4.50}$$

This has been compactified with periodic time such that $\tau \sim \tau + \frac{2\pi R}{r_+}$ and $\varphi \sim \varphi + 2\pi$. In the boundary limit as $r \to \infty$ we also obtain

$$\frac{\beta}{L} = \frac{R}{r_+} \ll 1. \tag{4.51}$$

As in the previous chapter's calculations, we define a subsystem A on the boundary, depicted in Figure 4.2, with boundary points at $\varphi = 0$ and $\varphi = \frac{2\pi l}{L}$. Perform the coordinate transformation:

$$r = r_+ \cosh(\rho), \quad \tau = \frac{R}{r_+}\theta, \quad \varphi = \frac{R}{r_+}t \tag{4.52}$$

then we obtain

$$ds^2 = R^2(\cosh^2(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)d\theta^2), \tag{4.53}$$

which appears to be the Euclidean global AdS metric in 3-dimensions. Taking a static τ surface implies that we set $d\theta = 0$,

$$ds^2 = R^2(d\rho^2 + \cosh^2(\rho)dt^2). \tag{4.54}$$

The arclength λ parametrisation condition (as justified earlier) gives us the equation

$$\dot{\rho}^2 + \cosh^2(\rho)\dot{t}^2 = 1. \tag{4.55}$$

We obtain the other require equation of motion using the Euler-Lagrange equations, starting with the Lagrangian

$$\mathcal{L} = \frac{1}{2}R^2(\dot{\rho}^2 + \cosh^2(\rho)\dot{t}^2). \tag{4.56}$$

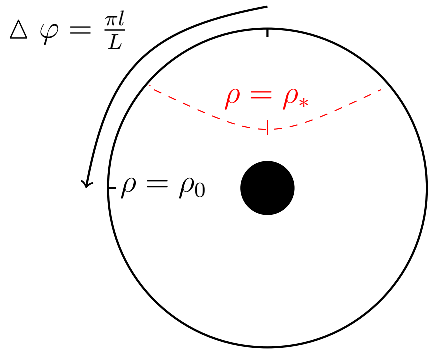

Figure 4.3: We specify boundary conditions for integration on the foliation of the BTZ black hole, note that due to symmetry we need only integrate over half of the minimal surface

Due to the lack of dependence on t,

$$\dot{t} = \frac{\alpha}{\cosh^2(\rho)}. \tag{4.57}$$

At the turning point of the minimal surface, we have $\dot{\rho} = 0 \implies \rho = \rho_*$ as shown in Figure 4.3. Substituting this into (4.55) we find that

$$\alpha = \cosh(\rho_*). \tag{4.58}$$

To compute the area element of the minimal surface we must first acquire some of the constituents to the inevitable integral that must be performed. Begin with

$$\frac{dt}{d\rho} = \frac{\dot{t}}{\dot{\rho}} = \frac{\alpha}{\cosh(\rho)\sqrt{\cosh^2(\rho) - \alpha^2}}. \tag{4.59}$$

For the purposes of a later integration, define $x = \frac{\alpha \sinh(\rho)}{\sqrt{\cosh^2(\rho) - \alpha^2}}$. Then

$$\begin{aligned}
\frac{dx}{d\rho} &= \frac{d}{d\rho}\left(\frac{\alpha \sinh(\rho)}{\sqrt{\cosh^2(\rho) - \alpha^2}}\right), \\
&= \frac{d}{d\rho}\left(\alpha \sinh(\rho)(\cosh^2(\rho) - \alpha^2)^{-\frac{1}{2}}\right), \\
&= \alpha \cosh(\rho)(\cosh^2(\rho) - \alpha^2)^{-\frac{1}{2}} - \alpha \sinh^2(\rho)\cosh(\rho)(\cosh^2(\rho) - \alpha^2)^{-\frac{3}{2}}, \\
&= \frac{-\alpha \cosh(\rho)}{\sqrt{\cosh^2(\rho) - \alpha^2}}\left(\frac{\alpha^2 - 1}{\cosh^2(\rho) - \alpha^2}\right).
\end{aligned} \tag{4.60}$$

We solve (4.59) by rearranging and integrating, which we will see makes use of this x substitution. The details are as follows

$$
\begin{aligned}
t &= \int d\rho \frac{\alpha}{\cosh(\rho)\sqrt{\cosh^2(\rho)-\alpha^2}}, \\
&= -\int dx \frac{\sqrt{\cosh^2(\rho)-\alpha^2}}{\alpha\cosh(\rho)} \left(\frac{\cosh^2(\rho)-\alpha^2}{\alpha^2-1}\right) \frac{\alpha}{\cosh(\rho)\sqrt{\cosh^2(\rho)-\alpha^2}}, \\
&= -\int dx \frac{\cosh^2(\rho)-\alpha^2}{(\alpha^2-1)\cosh^2(\rho)}, \\
&= -\int dx \frac{\cosh^2(\rho)-\alpha^2}{\alpha^2\sinh^2(\rho)+\alpha^2-\sinh^2(\rho)-1}, \\
&= -\int dx \frac{\cosh^2(\rho)-\alpha^2}{\alpha^2\sinh^2(\rho)+\alpha^2-\cosh^2(\rho)}, \\
&= \int dx \frac{1}{\left(\frac{-\alpha\sinh(\rho)}{\sqrt{\cosh^2(\rho)-\alpha^2}}\right)^2-1}, \\
&= -\int dx \frac{1}{x^2-1}, \\
&\implies t = -\mathrm{arccoth}(x) + C'.
\end{aligned}
\tag{4.61}
$$

We still need to determine the constant C', so we write

$$
\coth(-t-C') = \frac{\alpha\sinh(\rho)}{\sqrt{\cosh^2(\rho)-\alpha^2}}, \tag{4.62}
$$

and note that $t=0 \implies \coth(-C') \to \infty \implies C'=0$, giving the following equation

$$
\coth(-t) = \frac{\alpha\sinh(\rho)}{\sqrt{\cosh^2(\rho)-\alpha^2}}. \tag{4.63}
$$

If we take that after some $t = \triangle\, t$ we hit $\rho=\rho_0$ following the minimal surface to the boundary from its turning point, we can compute the following:

$$
\begin{aligned}
\coth(\triangle\, t) &= -\frac{\alpha\sinh(\rho)}{\sqrt{\cosh^2(\rho)-\alpha^2}}, \\
\implies \coth(\triangle\, t) &= -\alpha = -\cosh(\rho_*), \\
\implies \frac{\cosh(\triangle\, t)}{\sinh(\triangle\, t)} &= -\cosh(\rho_*), \\
\implies \sinh^2(\rho_*) &= \frac{\cosh^2(\triangle\, t)}{\sinh^2(\triangle\, t)} - 1 = \frac{1}{\sinh^2(\triangle\, t)}, \\
\implies \sinh(\triangle\, t) &= \frac{1}{\sinh(\rho_*)}.
\end{aligned}
\tag{4.64}
$$

where we have used the fact that $\cosh(\rho_0) \approx \sinh(\rho_0)$ in the $\rho_0 \to \infty$ limit and $\cosh(\rho_0) \gg \alpha$. Now we move to the computation of the area element of the minimal surface - simply a line element as we are in 3-dimensions.

$$L_{\gamma_A} = \int ds = 2R \int_0^{\Delta\lambda} d\lambda(\dot{\rho}^2 + \cosh^2(\rho)\dot{t}^2)^{frac12}, \tag{4.65}$$

and we can change variables such that $d\lambda = d\rho \frac{\cosh(\rho)}{\sqrt{\cosh^2(\rho)-\alpha^2}}$ to obtain

$$L_{\gamma_A} = 2R \int_{\rho_*}^{\rho_0} d\rho \frac{\cosh(\rho)}{\sqrt{\cosh^2(\rho) - \alpha^2}}. \tag{4.66}$$

We use the substitution $u = \sinh\rho \implies du = d\rho\cosh(\rho)$ and find that

$$L_{\gamma_A} = 2R \int_{u_*}^{u_0} du \frac{1}{\sqrt{1 + u^2 - \alpha^2}}. \tag{4.67}$$

And another substitution $v = \frac{u}{\sqrt{\alpha^2-1}}$, leading to

$$\begin{aligned}
L_{\gamma_A} =& 2R \int_{v_*}^{v_0} dv \frac{1}{\sqrt{v^2 - 1}}, \\
=& 2R \left[\operatorname{arccosh}\left(\frac{\sinh(\rho_0)}{\sqrt{\alpha^2 - 1}}\right) - \operatorname{arccosh}\left(\frac{\sinh(\rho_*)}{\sqrt{\alpha^2 - 1}}\right)\right], \\
=& 2R\log\left(\frac{\sinh(\rho_0)}{\sinh(\rho_*)} + \sqrt{\frac{\sinh^2(\rho_0)}{\sinh^2(\rho_*)} - 1}\right), \\
=& 2R\log\left(\frac{\sinh(\rho_0) + \sqrt{\sinh^2(\rho_0) - \sinh^2(\rho_*)}}{\sinh(\rho_*)}\right), \\
\approx& 2R\log\left(\frac{2\sinh(\rho_0)}{\sinh(\rho_*)}\right), \\
=& 2R\log\left(\frac{e^{\rho_0 - e^{\rho_0}}}{\sinh(\rho_*)}\right) \approx 2R\log\left(\frac{e^{\rho_0}}{\sinh(\rho_*)}\right), \\
=& 2R\log\left(\frac{\beta}{a}\sinh(\triangle\, t)\right), \\
=& 2R\log\left(\frac{\beta}{a}\sinh\left(\frac{\pi l}{\beta}\right)\right).
\end{aligned} \tag{4.68}$$

We have made use of $\rho_0 \to \infty$, the fact that $\triangle\, t = \frac{r_+}{R} \triangle\, \varphi = \frac{r_+}{R}\frac{\pi l}{L}$ and (4.51), implying that $\triangle\, t = \frac{\pi l}{\beta}$.

The entanglement entropy is then

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{a}\sinh\left(\frac{\pi l}{\beta}\right)\right). \tag{4.69}$$

This reproduces the exact known result for a 2-dimensional CFT with finite temperature [10]. We see from this that we can utilise the AdS/CFT correspondence in spacetimes containing black holes in such a way that known results are exactly identified. This presents a strong case for the use of the AdS/CFT correspondence to probe black hole interiors using physics on the boundary (horizon surface).

Chapter 5

Information, Entropy, and Emerging Islands

In this chapter we bring together all of the previous discussion and look to apply it to a puzzle that has remained unsolved since the 1970's. We will introduce the information paradox as initially stated by Hawking [32] and with some additional commentary from [9] to introduce the concept of Hawking radiation. We will then consider the concept of quantum extremal surfaces and their role in the paradox considering recent work by Almheiri, Engelhardt, Marolf, and Maxwell [33]. Then we will conclude the chapter with discussion on the Islands Conjecture [34].

5.1 The Hawking Information Paradox

Entropy - particularly entanglement entropy - is quite apparently the cornerstone of our discussion here, and so it is important to ensure that we have defined the different types of entropy quite clearly. An important distinction to make when doing so is defining fine- versus coarse-grained entropy. Fine-grained entropy has actually been seen already (3.46), but we also introduce coarse-grained entropy. To compute the coarse-grained entropy of a system we first take all density matrices that give the same result as the system for tracked observables. Say we have $\tilde{\rho}$ denoting all of the density matrices we wish to consider, for each of these we compute their von Neumann entropy $S(\tilde{\rho})$ and then we maximise the calculated entropy over all possible $\tilde{\rho}$. Note that these entropies obey the second law of thermodynamics. The primary example of coarse-grained entropy we shall use is the Bekenstein-Hawking entropy (3.62).

For a fine-grained entropy of quantum fields we can (similar to the foliation in our holographic entanglement entropy calculations) define this for a fixed time foliation corresponding to a spatial region Σ. The fine-grained entropy is

$$S_{vN}(\Sigma) \equiv S_{vN}(\rho_\Sigma), \tag{5.1}$$

where ρ_Σ is the density matrix for the specified region. For the purposes of the information paradox we will be considering a semiclassical approximation of gravity, and so - in keeping with Almeiri et al [9], with the new semiclassical entropy denoted

$$S_{semi}(\Sigma) \tag{5.2}$$

which describes the fine-grained or von Neumann entropy of our quantum fields, which are defined on a classical geometry.

As for the paradox itself, there are a few ways of thinking about it. Almheiri et al [9] begins with a black hole that emits thermal radiation formed from the collapse of a pure state. The thermal behaviour experienced we know to emerge from dividing our space into an interior and an exterior, and some presence of short entanglement between the states in each of these subregions. In this case, even if overall we have a pure state, observers in either of the subregions will see a mixed state (as the other state from the entangled pair is inaccessible). Now this does not violate any laws of physics as presented, however when one considers that black holes do in fact evaporate, this causes issues. We see the interior region shrinking as the exterior builds, but eventually when the black hole evaporates this will stop. Say we have taken our interior region to have become infinitesimally small - the states on the exterior still appear like mixed states entangled to inaccessible particles on the interior. But if the black hole evaporates completely, we surely must have a pure state again? Well, this would certainly be convenient, but the laws we have obeyed thus far describe that the exterior should continue to be a mixed state.

What is the issue here? Unitarity. If I start in a pure state and time evolution is unitary then I expect to end in a pure state. Clearly with the evaporating black hole, this was not the case; it is not reasonable to suspect that something odd is happening with the entanglement entropy at this point, so let's track that. At early times the black hole essentially only has thermal radiation as the radiation in the exterior is entangled with the black hole quantum system - the entropy of the radiation is rising. As the black hole continues to evaporate, it shrinks and so we see the thermodynamic entropy of the black hole decreasing, which is trivially seen from the Bekenstein-Hawking entropy formula (3.62). But this is where we start to encounter problems - the black hole is still radiating and so the entropy of radiation on the exterior rises. So we have some contradiction here in that the black hole began as a pure state, but we end up with a final state of highly entangled radiation.

It seems logical that we would expect the entropy of the radiation to decrease at a certain point with the black hole entropy, and this is exactly what was proposed by Page [25], [26]. Hawking argued that one could not have a unitary map from the initial pure state shell to the mixed state radiation as it implies that many initial states could result in the same final state. This is exactly what was being put forward by Page in that there is no way for the Page curve to go back to $S_{Rad} = 0$. In other words, the entropy of radiation cannot follow the Hawking curve which leads to this paradoxical behaviour, we should instead look for solutions that adhere to the Page curve. The Page curve proposes that at a given time when $S_{BH} = S_{Rad}$ - which occurs at the Page time - there must be some mechanism that causes the entropy of radiation to then decrease.

5.2 Emerging Islands

The notion of emerging islands as a result of a quantum extremal surface is one of several generalisations of the original RT formula we have used throughout our discussion thus far [8],

[10]. To reach this concept, we must first provide a brief overview of an initial generalisation of the RT formula proposed by Hubeny, Rangamani and Takayanagi (HRT) [36] which allows for the holographic calculation of entanglement entropy in a temporally varying system. We will see that emerging islands within the interior of the black hole could help to reproduce the Page curve, in the appropriate setting.

5.2.1 HRT Surfaces

Hubeny et al [36] put forward a covariant entanglement entropy proposal as a generalisation to the RT formula, starting with the typical AdS/CFT setup and using Poincaré coordinates for convenience. At some time t we divide the boundary into two regions A_t and $B_t = A_t^c$ and denote the boundary between the two ∂A_t - this is a spacelike surface on the boundary of the entire asymptotically AdS spacetime with dimension $d-2$.

One can construct the spacelike surface ∂A_t via light-sheets as outlined in [36]. In short, the intersection of upper and lower light sheets creates a spacelike surface that extends into the bulk, call this $\mathcal{Y}_t = L_t^+ \cup L_t^-$. The proposed correction to the RT formula given this set up is

$$S_{A_t} = \frac{\min_{\mathcal{Y}}(\text{Area}(\mathcal{Y}_t))}{4G_N^{(d+1)}} = \frac{\text{Area}(\mathcal{Y}_{A_t}^{\min})}{4G_N^{(d+1)}}, \tag{5.3}$$

where we have called the minimal area surface $\mathcal{Y}_{A_t}^{\min}$. Now, whilst this is manifestly covariant in that we successfully encode time dependence, it is quite a lengthy process to construct each of the light-sheets in the bulk that are feasible. This can be vastly streamlined with a second proposal.

The claim is that - in the same setup as above - we have some region $\mathcal{A}$ and can find the holographic entanglement entropy to be

$$S_{\mathcal{A}} = \frac{\text{Area}(\mathcal{Y}_{ext})}{4G_N^{(d+1)}} \tag{5.4}$$

Our surface $\mathcal{Y}_{ext}$ is codimension two surface such that $\partial\mathcal{Y} = \partial\mathcal{A}$ and that has zero null geodesic expansions. For cases where there are many of these surfaces, we choose the one with minimum area. Computations are provided by [36] that replicate the results of entanglement entropy for CFTs using the holographic prescription in the bulk, but we don't dive any further into these here.

5.2.2 Quantum Extremal Surfaces

This is a good place to take stock of what we have encountered thus far, before we introduce some more abstract notions. We know that black hole thermodynamics gives us an entropy consistent with the second law of thermodynamics. Quantum corrections are needed for this as - due to Hawking radiation - black holes are intrinsically quantum systems. We can write the semi-classical generalised entropy of the black hole horizon H as the area term (entropy of the black hole), plus the entropy in the exterior and the counterterms:

$$S_{gen}(H) = \frac{\text{Area}(H)}{4G_N} + S_{out} + \text{counterterms}, \tag{5.5}$$

with S_{out} defined as the von Neumann entropy for the state of the matter fields outside of the black hole (including Hawking radiation). The counterterms are something we saw in chapter 4 and similarly, they absorb divergences in S_{out} and subleading divergences on the horizon.

In the AdS/CFT arena, we have seen the Ryu and Takayanagi entropy-area relation which relates the area of a minimal surface extending from the boundary region into the bulk. In the appropriate classical limit, it has been proposed by HRT that the entanglement entropy for a patch of the boundary R with a CFT living on it can be computed from the gravitational theory using a codimension two extremal surface X such that $\partial X = \partial R$ with X homologous to R. We can see why the two theories are valid when you take the spacetime to be static in the extremal surface case - in the case where a time foliation is preferred (we choose a static time slice), the extremal surface coincides with the minimal surface on a constant time slice, as seen in [8].

Following this, there was work by Faulkner, Lewkowycz, Maldacena (FLM) [37] that computed the quantum corrections to the HRT entanglement entropy proposal:

$$S_R = \frac{\text{Area}(X)}{4G_N} + S_{ent} + \text{counterterms} = S_{gen}(X), \tag{5.6}$$

with the bulk entanglement entropy being denoted S_{ent}. We note that this is strikingly similar to (5.5). The FLM prescription essentially generalises the classical prescription (RT/HRT) by including quantum corrections to the Euclidean gravitational path integral.

Engelhardt and Wall [35] then define an entangling surface E as any codimension two spacelike surface that divides a Cauchy surface into two pieces - a cut - to define an interior and exterior on the Cauchy slice. $S_{out}(E)$ would now be the entanglement entropy in the exterior of E and $S_{gen}(E)$ we take as the black hole entropy but on E rather than a slice of the horizon. One can also make a unitarity argument that the choice of Cauchy slice passing through E (our cut) doesn't change the generalised entropy $S_{gen}(E)$.

The crucial difference between the prescriptions set out by FLM [37] and Engelhardt and Wall (EW) [35] is that instead of extremising for area and then adding S_{out} (FLM regime), we should in fact extremise the total generalised entropy S_{gen}. The following conjecture was put forward by EW:

"The entanglement entropy of a region R in a field theory with a holographic dual is given at any order in $\hbar$ in the holographic dual by the generalised entropy of the quantum extremal surface $\mathcal{X}_R$ anchored at R, homologous to R"

$$S_R = S_{gen}(\mathcal{X}_R). \tag{5.7}$$

After seeing the work by FLM and EW, we can see how the notion of QES can be applied to the black hole to reproduce the Page curve. In [35, 36, 37] we search for an extremal surface that minimises spatially whilst maximises temporally. As summarised by [9], one first chooses a Cauchy (spatial) slice and proceeds to find the minimal surface, then it is a case of maximising over the choice of Cauchy slice taken. More specifically, for a codimension two surface X

$$S = \min_X \left\{ \mathrm{ext}_X \left[\frac{\mathrm{Area(X)}}{4G_N} + S_{semi}(\Sigma_X) \right] \right\}, \tag{5.8}$$

where Σ_X is the region bounded by X and the cut off, and the final semiclassical entropy term is the von Neumann entropy of quantum fields on Σ_X. Then the generalised entropy is simply

$$S_{gen}(X) = \frac{\mathrm{Area(X)}}{4G_N} + S_{semi}(\Sigma_X). \tag{5.9}$$

This concept is particularly dependent on the geometry of the black hole interior, which may vary regardless of how similar the exterior appears. If the interior geometry differs we expect the fine-grain entropy to differ also. What we are really doing is choosing some Cauchy time slice outside of the black hole and moving into the interior to find its minimum. It is certainly possible that the surface within the exterior could completely shrink, not giving any area contribution.

One can consider an evaporating black hole, and study how the entropy would behave throughout the stages of this process. We assume that we have started the collapsing shell process for the black hole in a pure state. At very early times, we can consider the scenario prior to any evaporation or any Hawking radiation being released. For each of the Cauchy slices at this time, we cannot find an extremal surface by deforming along the slice into the black hole interior. This is the *vanishing extremal surface*. The early evaporation time gives a slightly different picture however.

Early Evaporation:

The von Neumann entropy is no longer zero due to the entanglement between the Hawking radiation emitted and the interior Hawking quanta. As the black hole evaporates, the entropy seems to continue increasing. However, there is now a *non-vanishing extremal surface* that appears shortly after the Hawking radiation starts to escape - this is shown explicitly in [42, 43].

Say we have chosen to locate the extremal surface at a time t during early evaporation. To determine the precise location of the extremal surface following the method of [9], we move back along the cutoff surface by a time of order $r_s \log(S_{BH})$ (the "scrambling time") and we shoot an ingoing light ray. It turns out that the extremal surface lies near the point of intersection between the light ray and the horizon.

For this case, the generalised entropy is approximately the Bekenstein-Hawking entropy (3.62). There has not been a great deal of Hawking radiation yet, so the extremal surface CFT has a relatively low entropy so we can neglect the semiclassical entropy contribution. The generalised entropy is decreasing here as the black hole surface area is evaporating.

Consider starting with the extremal surface X by the horizon. As we move X inwards the entropy S_{semi} is decreasing because the new Hawking radiation modes are purifying the outgoing quanta already included in the outside region. Having completed this purification, moving X further inwards begins to include modes that are already entangled to "outside"

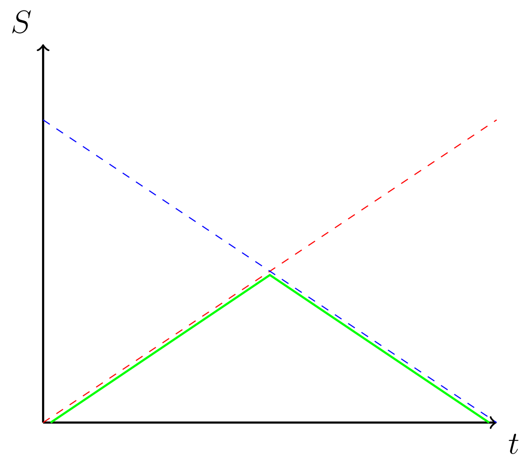

Figure 5.1: Displaying entropy S over time t, we have the non-vanishing surface represented by the blue dashed line, the vanishing surface represented by the red dashed line, and the Page curve for fine-grained entropy shown in green.

modes in the exterior, bringing up the value of S_{semi}. This is exactly what is shown by Almheiri et al [34] to balance out the change in area.

We have found two surfaces here, the one at early times before Hawking radiation has occurred is the vanishing surface and the non-vanishing surface being the decreasing surface just discussed. During the early time period we only have this vanishing surface which provides a contribution to the entropy which starts at zero and grows monotonically as the black hole evaporates. After some time the non-vanishing surface appears when we have at least had some small amounts of Hawking radiation - this decreases in entropy as the black hole evaporates. The key point here is that our corrected entropy formula (5.6) [35, 37] is selecting the minimum of the extremal surfaces. Initially the vanishing surface is minimal, but this has increasing entropy and the non-vanishing surface has decreasing entropy - at some time t_P we expect the minimum extremal surface to switch to the non-vanishing surface. In other words, for time $t < t_P$ the fine-grained entropy is defined by the vanishing surface and for time $t > t_P$ the fine-grained entropy is defined by the non-vanishing surface. As the first is increasing in entropy from zero and the second is decreasing in entropy to zero, we see that we reproduce the Page curve and t_P can be taken as the Page time as shown in Figure 5.1. This is an outline of how, for the black hole, one can use the gravitational entropy formula (5.8) to compute the fine-grained entropy and show it accurately follows the Page curve. An explicit calculation of this can be found by Almheiri et al in [33] which considers another mechanism that sheds light on a new perspective one can consider the information paradox from, which also speaks to the content of the next subsection.

5.2.3 The Islands Conjecture

To study the classical information paradox in AdS spacetime is actually quite complex, even with a tool as powerful as the AdS/CFT correspondence. It is widely known that photons in the AdS cylinder reach the boundary $r = \infty$ in finite time and are forced to bounce off of the horizon back into the spacetime (details can be found in [44]).

Now imagine we have a black hole in the AdS spacetime. This black hole will never be able to evaporate as the radiation will just bounce back into the black hole given enough time has passed. One way to study an evaporating black hole in this setup is to couple the AdS space to an auxiliary system. Coupling to an auxiliary system with boundary conditions that allow for the exchange of stress-energy means the radiation will move from AdS into the auxiliary system rather than bouncing back into the black hole - this allows evaporation in AdS to occur. We can take this auxiliary system to be Minkowski space without gravity.

Almheiri et al [34] consider this exact problem in 2-dimensional JT (Jackiw-Teitelboim) gravity - this is just typical Einstein-Hilbert gravity with an additional dilaton field ϕ,

$$I_g[g_{ij}, \phi] = \int d^2y\sqrt{-g}\left(\frac{1}{16\pi G_N^{(2)}}\phi R + U(\phi)\right). \tag{5.10}$$

We want to consider the 2-dimensional gravity theory with the addition of holographic matter. We “add matter” by taking this to be a CFT 2 dimensions with some fields denoted χ, giving the action

$$I[g_{ij}, \phi, \chi] = I_g[g_{ij}, \phi] + I_{CFT}[g_{ij}, \chi]. \tag{5.11}$$

Now we employ the AdS/CFT tools we have explored in the previous sections - the CFT has a 3-dimensional holographic dual that will be crucial to this notion of islands. More specifically, the 3-dimensional theory with boundary metric g_{ij} is dual to a CFT in 2 dimensions described by $I_{CFT}[g_{ij}, \chi]$. To ensure the semi-classical limit is appropriate and that the dual space is of large radius we require both that for central charge c (we are in 2 dimensions) we have $1 \ll c \ll \frac{\phi}{4G_N^{(2)}}$. For there to be an Einstein gravity dual also we will require that the CFT is strongly coupled - the gravity dual will be important in considering how the radiation connects to the interior in the higher-dimensional dual perspective.

We know from AdS/CFT that we can find a gravity dual for the auxiliary system, but we need to find a 3-dimensional dual for the entire system. We do this by adding a scalar ϕ to the above geometry that lives on the boundary with action I_g and integrating over both the scalar field ϕ and the metric g_{ij}. This is the Randall-Sundrum model [45] and produces a dynamical boundary brane called the Planck brane. This brane is embedded in AdS_3 and allows us to find a holographic dual for the full action (5.11). The reason we want this dynamical boundary is so that we can project the CFT bath on the half line and the Planck brane holographically as one system, allowing us to glean more information about how the higher dimensional dual is connected - something that is crucial to the island conjecture.

We can allow the black hole to evaporate into a coupled CFT bath defined on the half-line which we take to be the same CFT as the matter CFT in (5.11), leading to a set up akin to Figure 5.2. This contains a precursor detailed by [34] in which we assume that it is reasonable to neglect backreaction effects and can think of the matter as living on a fixed, non-dynamical background. We also assume the boundary allows for a free exchange of stress-energy.

We get this combined system now that Figure 5.2 depicts and can write it in two descriptions:

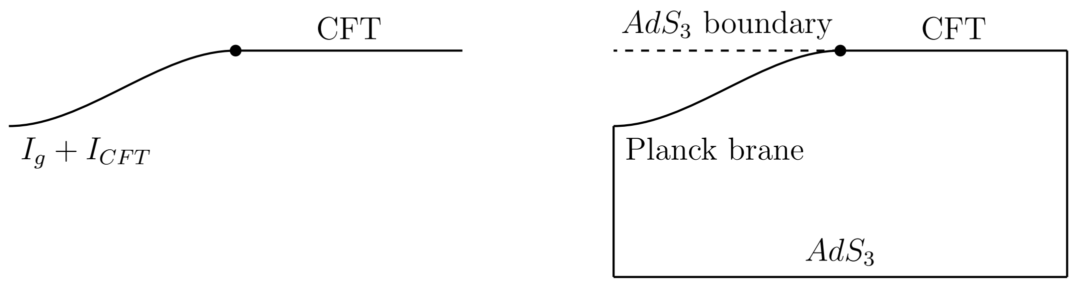

Figure 5.2: Left: 2D gravity view in which the dot represents the boundary between the gravity theory and the CFT, and the gravity theory line contains the full action. Right: 3D gravity holographic dual of the full 2D picture, the Planck brane descends to the left from the boundary point (below the AdS_3 boundary.

- **2D Gravity:** the 2-dimensional gravity-plus-matter theory living on the left of the boundary point is coupled to a 2-dimensional field theory living on the right.
- **3D Gravity:** the 3-dimensional gravity theory in AdS_3 with a dynamical boundary (the Planck brane) on the left of the boundary point, with a rigid boundary on the right of the boundary point.

In the 2D Gravity description above, we compute the fine-grained entropy by extremising S_{gen} following the EW prescription

$$S_{gen}(y) = \frac{\phi(y)}{4G_N} + S_{bulk+}[\mathcal{I}_y], \tag{5.12}$$

where $S_{bulk+}[\mathcal{I}_y]$ is the bulk von Neumann entropy of the interval $\mathcal{I}_y$, and y is a point in the bulk. We extremise over y and then take the minimum over the extrema. Noting that we are in 2-dimensions, the quantum extremal "surface" is actually the resulting point (y_e^+, y_e^-). $\phi(y)$ is taken to be the "area" of the point - the coefficient of the curvature term in (5.10).

The 2-dimensional CFT contributes to the entropy from the matter fields χ in S_{bulk+} and has a holographic dual. This means we can compute this contribution using RT/HRT - we would have to find a minimal/extremal surface in the 3-dimensional geometry. This implies that the extremisation of S_{gen} in 2-dimensions is equivalent to the RT/HRT area extremisation in 3-dimensions containing a dynamical brane.

Entanglement Wedge of the Black Hole

At late times the QES lies behind the horizon at the point (y_e^+, y_e^-). A past-directed light ray would reach the AdS_2 boundary at a scrambling time less than t, at which time we can compute the entropy

$$y_e^+ = t = \frac{1}{2\pi T(t)} \log\left(\frac{S_{BH}(T(t)) - S_0}{c}\right) + ..., \tag{5.13}$$

where $T(t)$ is the black hole temperature at time t and S_0 is the extremal entropy.

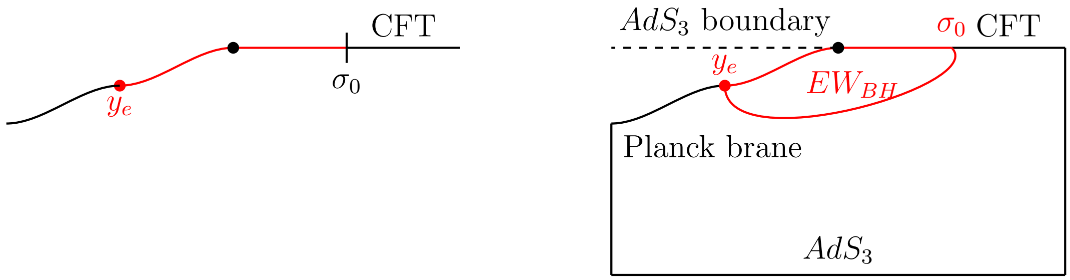

Figure 5.3: Left: 2D gravity view of the black hole entanglement wedge. Right: 3D gravity holographic dual of the full 2D black hole entanglement wedge denoted EW_{BH}.

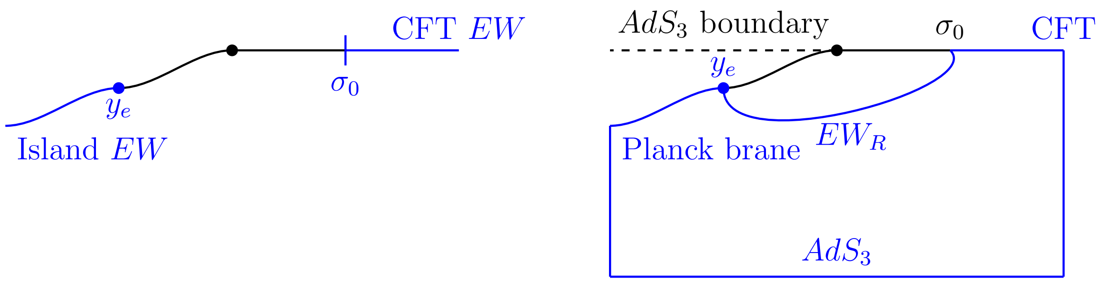

Figure 5.4: Left: 2D gravity view of the radiation entanglement wedge - this comprises of two wedges (CFT and island) that appear to be disconnected. Right: 3D gravity holographic dual of the full 2D radiation entanglement wedge denoted EW_R which connected the island to the exterior radiation.

We can define the *entanglement wedge* (EW) as the causal domain of a spacelike slice from (t, σ_0) to (y_e^+, y_e^-), where σ_0 denotes the end of a small interval into the CFT bath. This implies that in our 2-dimensional model, the bulk entanglement entropy computation only involved the calculation on an interval. This is more easily visualised when considering Figure 5.3. Almheiri et al [34] describe how this leads to the black hole entropy

$$S_{\text{Black Hole}}(t) = S_{BH}(T) + \log \text{ terms}. \tag{5.14}$$

S_{BH} denotes the usual Bekenstein-Hawking entropy and the log terms are those that are logarithmic in the Bekenstein-Hawking entropy of the initial state. This is a decreasing entropy as the temperature decreases with time.

Entanglement Wedge of the Radiation

The entropy of the radiation here is just the entropy of the state in the CFT bath that lies past the point σ_0 we considered in the previous section. The naive computation involves taking the entire slice and tracing out anything that doesn't lie within the desired region - this reproduces the Hawking entropy curve. Since the entanglement wedge EW_{BH} covers some of the interior we can take the rest of the interior as belonging to the EW_R - the radiation entanglement wedge.

The incredible thing about the AdS/CFT correspondence we have made use of is now

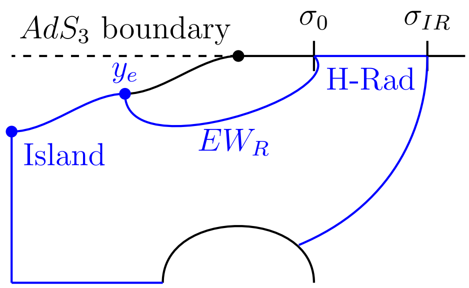

Figure 5.5: A more precise view of the "bridge" connecting the island to the Hawking radiation (H-Rad). This wormhole behaviour is apparent only in the higher dimensional holographic projection.

manifest in Figure 5.4. In the 2-dimensional depiction we appear to have broken down the EW_R into two disconnected components - the "island" entanglement wedge EW_I and the CFT entanglement wedge EW_{CFT}. However, when we consider the holographic dual of this, we notice that the island and the exterior radiation appear to be connected, giving just one complete entanglement wedge for the radiation EW_R. There is a "bridge" of sorts - often referred to as a *wormhole* - connecting the island to the exterior radiation once we consider this higher dimensional holographic projection. One can think of this in the spirit of ER=EPR as proposed by Susskind [46], in short this idea is that Hawking radiation and the interior of the black hole via a wormhole because of their entanglement to each other.

Conjecture: The Islands Rule

A more precise description of this is given by [34], the conditions are that we begin with a low energy black hole state, the radiation sits within a finite interval $[\sigma_0, \sigma_{IR}$ (σ_{IR} is large enough to hold all Hawking radiation), and we began with a Cardy conformal boundary condition [47] where we have the coupling between the CFT bath and the black hole spacetime. This will give a more accurate 3-dimensional depiction of the entanglement wedge of the radiation in which the wormhole concept is more apparent.

Considering Figure 5.5, we are nudged along the line of thinking that one must include the interior in the holographic calculation of the radiation entropy. Hence, Almheiri, Mahajan, Maldacena and Zhao stated a new rule [33] for the computation of entropies in effective gravity theories:

"If we consider a state in a quantum field theory that is entangled with some other system that lives in a gravity theory, then we should use the RT/HRT/EW method and include all possible "islands" that could extremize the entropy."

The entropy of some region A in a QFt can be computed as

$$S(A) = \text{Min}_{\mathcal{I}_g}\left[\text{Ext}_{\mathcal{I}_g}\left(S_{eff}(A \cup \mathcal{I}_g) + \frac{\text{Area}(\partial \mathcal{I}_g)}{4G_N}\right)\right], \tag{5.15}$$

where we extremise over all islands $\mathcal{I}_g$, then minimise over the obtained extrema. Note that $\text{Area}(\partial\mathcal{I}_g)$ is the area of the island boundary and that S_{eff} is the semi-classical gravitational entropy. The islands $\mathcal{I}_g$ are *quantum extremal islands*.

Conclusion

We began with an overview of the AdS/CFT correspondence, starting with the basics of conformal field theory and anti-de Sitter spacetime - the main result of this is, in essence, that for any $d+1$-dimensional gravity theory in AdS one can always find a dual d-dimensional CFT. With appropriate boundary conditions this essentially tells us that the gravitational path integral is equivalent to the CFT path integral.

Application of the correspondence follows very naturally from this, and we focus on one application in particular - the holographic derivation of entanglement entropy. We derive some of the main results given by Ryu and Takayanagi [8] using the RT formula, before giving some background on black holes, their thermodynamics, and how we can apply the RT formula to black holes in AdS space.

Once we have reviewed these tools, it is unsurprising that a central discussion point would be the information paradox, and how the possible resolutions to this problem have evolved as the generalisations of the original RT formula have evolved. Of course there are many more contexts in which we can probe the utility of the AdS/CFT correspondence and holographic entanglement entropy, but our objective here was to see how the story of the iterations of the RT formula spurred on developments in the information paradox arena.

The RT (Ryu, Takayanagi) formula [8] tells us that one can find the entanglement entropy holographically for a d-dimensional CFT from the area of $d-1$-dimensional minimal surfaces at fixed time in AdS_{d+1}. The HRT (Hubeny, Rangamani, Takayanagi) formula [36] generalises this to include time dependence. This is all very classical at the moment however, so FLM (Faulkner, Lewkowycz, and Maldacena) [37] derive the quantum corrections for the HRT formula, this is what really opened doors for entanglement entropy calculations when considering the black hole as a quantum system. Engelhardt and Wall do exactly this in the quantum extremal surface conjecture [35] noting that one should extremise the generalised entropy rather than the extremised area. Finally, this led to the islands rule posed by Almheiri et al [34] stating that we should include all possible islands in our consideration of extremal surfaces that extremise entropy. This is truly insightful, making use of holography to see that a island - seemingly disconnected from Hawking radiation in the exterior - is actually connected to exterior radiation through an additional dimension when we consider the holographic dual.

In conjunction with these developments, we see the progress of the information paradox since it came into being in the 1970's when Hawking showed that black holes must evaporate. The particular context that is useful for framing the information paradox here is the Page curve. The entropy of a black hole should obey the Page curve, in that it initially increases, reaches a turning point at the Page time, and then monotonically decreases to zero by the time the black hole evaporates. The initial calculation of the black hole entanglement entropy by Hawking showed the entropy was monotonically increasing throughout, violating the unitarity of the black hole evaporation (the idea that we should not lose quantum information - if a black hole is initially in a pure state it should end in a pure state, but if there is entangled radiation after evaporation how is this possible?).

It turns out some answers to replicating the Page curve came through this use of holographic entanglement entropy. Almheiri, Engelhardt, Marolf, and Maxwell [33] and Penington [49] showed that for a black hole in AdS space coupled to some auxiliary system (similar to the previous chapter), there is an "entropic" phase transition that takes place in the minimal QES. This occurs exactly at the Page time and moves our extremal surface from one that is increasing in entropy to one that is decreasing, replicating Figure 5.1. This was a direct result of the conjectured "island rule". A detailed summary of this story can be found in [9].

A Note on Replica Wormholes

A recent and promising development is the use of replica wormholes to derive the fine-grained entropy for a black hole using the island conjecture. This is briefly outlined in [9] and computed explicitly in [42, 43].

Assume that the black hole forms from an initial pure state, looking to find the entropy of the final radiation. If we take the density matrix to be $\rho = |\psi\rangle \langle\psi|$ where ψ is the initial state, we can compute the von Neumann entropy using the replica trick. What this means is that $Tr[\rho^n]$ can be used to measure the purity of the state, but it can also be viewed as n copies (replicas) of the original system.

We consider n copies of the original theory and connect these replicas in a cyclic way. Say we want to take $n = 2$ for our black hole state, we would use a path integral approach and connect the exteriors of each black hole copy to find $Tr[\rho^2]$. The crucial point that we must keep in mind is similar to the rule for islands. For the gravitational path integral, we should sum over all possible topologies that contribute any gravity effects. What this implies is that we should take the gravitational path integral computation to include all possible ways in which the interiors of our black hole copies can be connected.

For the case of $n = 2$, we can either have the Hawking saddle in which the interiors of each replica don't connect to each other, or the replica wormhole saddle in which the interiors do in fact connect in a cyclic way. Whilst each replica in Figure 5.6 looks like two black holes, each is in fact just a representation of one black hole drawn in such a way that we see the interior as two throats connected through entanglement. This case will give the increasing entropy that creates the Hawking curve.

Because we know that we should count all possible topologies we also have to count the configuration in Figure 5.7 - this is the one in which the interiors of the replicas are connected by a wormhole. We call this the *replica wormhole*. When computing the entanglement entropy in this way we get the answer from the quantum extremal surface in the limit of $n \to 1$. There is a turning point at the Page time from the Hawking saddle to the replica wormhole saddle which reproduces the Page curve, we can see this more explicitly in [42, 43].

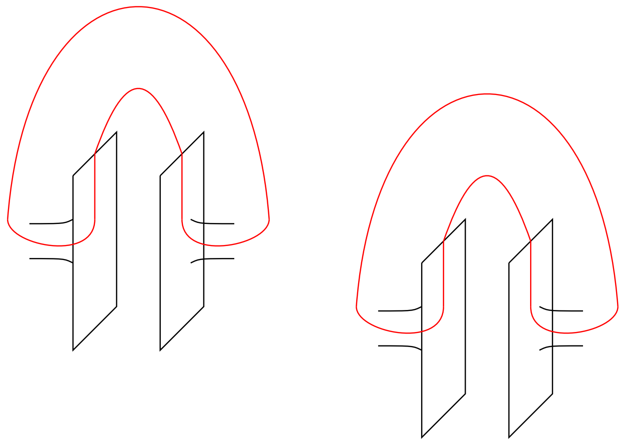

Figure 5.6: A heuristic diagram of the Hawking saddle to be counted in the gravitational path integral when we use the replica trick for $n = 2$. Red lines represent entanglement. The slices in the middle of each copy are the exterior and the interiors are actually just a single interior of the black hole as they are connected via entanglement. We have two disjoint black hole replicas.

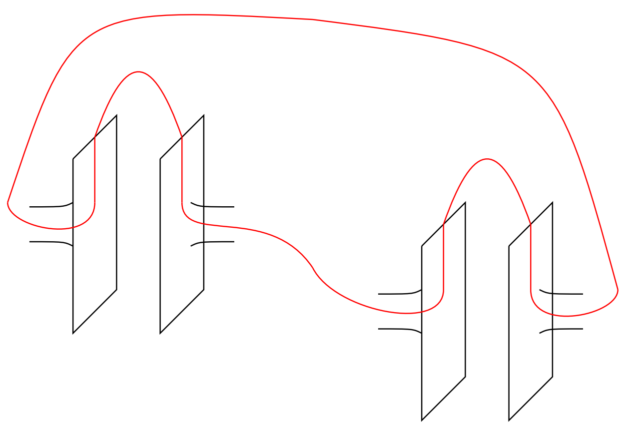

Figure 5.7: A heuristic diagram of the Replica wormhole saddle to be counted in the gravitational path integral when we use the replica trick for $n = 2$. Red lines represent entanglement. This time, the two replicas seem to be connected in a cyclic way, and we call the connection of the interiors to each other the replica wormhole.

Bibliography

[1] M. E. Peskin, D. V. Schroeder, *An Introduction to Quantum Field Theory*, first published by Westview Press, 1995. Published by CRC Press, Taylor & Francis Group, FL, 2018

[2] O. Aharony (a), S. Gubser (b), J. Maldacena (c), H. Ooguri (d,e), Y. Oz (f), *Large N Field Theories, String Theory and Gravity*, (a) Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA (b) Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA (c) School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540 (d) Department of Physics, University of California, Berkeley, CA 94720-7300, USA (e) Lawrence Berkeley National Laboratory, MS 50A-5101, Berkeley, CA 94720, USA (f) Theory Division, CERN, CH-1211, Geneva 23, Switzerland; 1999

[3] M. Luscher, G. Mack, *Global Conformal Invariance in Quantum Field Theory*, Institut fur Theoretische Physik der Universitat Bern, Bern, Switzerland, Comm. Math. Phys. 41 (1975) 203

[4] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity.* Pearson Education (publishing as Addison Wesley), San Francisco, CA, 2004.

[5] W. Nahm, *Supersymmetries and their representations*, CERN, Geneva, Switzerland, Nuclear Physics B135 149-166, 1978.

[6] J. Maldacena, *The gauge/gravity duality*, Institute for Advanced Study, Princeton, NJ 08540, USA, 2014.

[7] A. Lewkowycz (a), J Maldacena (b), *Generalised gravitational entropy*, (a) Department of Physics, Princeton University, Princeton, NJ, (b) School of Natural Sciences, Institute for Advanced Study, Princeton, NJ, 2013.

[8] S. Ryu, T. Takayanagi, *Holographic Derivation of Entanglement Entropy from AdS/CFT*, Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA, 2006.

[9] A. Almheiri (a), T. Hartman (b), J. Maldacena (a), E. Shaghoulian (b), A. Tajdini (b), *The entropy of Hawking radiation*, Institute of Advanced Study, Princeton, NJ, (b) Department of Physics, Cornell University, Ithaca, NY, 2020.

[10] S. Ryu (a), T. Takayanagi (b), *Aspects of Holographic Entanglement Entropy*, (a) Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA, (b) Department of Physics, Kyoto University, Kyoto 606-8502, Japan, 2006.

[11] G. Horowitz (Ed.) *Black Holes in Higher Dimensions*, Cambridge: Cambridge University Press. doi:10.1017/CBO9781139004176, 2012.

[12] G. 't Hooft, *A Planar Diagram Theory for Strong Interactions*, Nuclear Physics B72 (461), CERN, Geneva 1974.

[13] B. A. Dobrescu, *Beyond the Standard Model*, Theoretical Physics Department, Fermilab, Batavia, IL, USA, 2014.

[14] J. Polchinski, *String Theory Volume II: Superstring Theory and Beyond*, Institute for Theoretical Physics, University of California at Santa Barbara, Cambridge University Press, New York, 1998.

[15] J. D. Bekenstein, *Black Holes and Entropy*, Joseph Henry Laboratories, Princeton University, Princeton, Neu Jersey OS540 and Center for Relativity Theory, The University of Texas at Austin, Austin, Texas 78712 - Physical Review D 7.8, 1973.

[16] S. W. Hawking, *Black Holes and Thermodynamics*, California Institute of Technology, Pasadena, California 91125 and Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England, (Physical Review D, 13(2)), 1976.

[17] A Strominger (a), C Vafa (b), *Microscopic Origin of the Bekenstein-Hawking Entropy*, (a) Department of Physics, University of California Santa Barabara, CA, (b) Lyman Laboratory of Physics, Harvard University, Cambridge MA, Phys. Lett. B 379, 99, 1996.

[18] L. Susskind, J. Lindesay, *An Introduction to Black Holes, Information and the String Theory Revolution: The Holographic Universe*. World Scientific Publishing Co. Pte. Ltd., Singapore, 2005.

[19] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*. The University of Chicago Press, Chicago 60637, 1994.

[20] J. Polchinski, *The Black Hole Information Problem*. Kavli Institute for Theoretical Physics University of California Santa Barbara, CA 93106-4030, 2016

[21] W. G. Unruh, *Notes on black-hole evaporation*. Department ofPhysics, Miller Institute for Basic Research, University of California, Berkeley, California 94720 and Department of Applied Mathematics, McMaster University, Hamilton, Ontano, Canada, 1976.

[22] R. Tolman, *On the weight of heat and thermal equilibrium in general relativity*. Norman Bridge Laboratory, California Institue of Technology, Pasadena, California, Phys. Rev. 35, 904–924, 1930.

[23] E. Witten, *Anti-de Sitter Space, Thermal Phase Transition, And Confinement In Gauge Theories*. School of Natural Sciences, Institute for Advanced Study Olden Lane, Princeton, NJ 08540, USA, 1998.

[24] J. Maldacena, *Eternal black holes in Anti-de Sitter*. Jefferson Physical Laboratory Harvard University Cambridge, MA 02138, USA, 2003.

[25] D. Page, *Information in Black Hole Radiation*. CIAR Cosmology Program, Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2J1, 1993.

[26] D. Page, *Time Dependence of Hawking Radiation Entropy*. CIAR Cosmology Program, Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2J1, 2013.

[27] D. Harlow, *Jerusalem Lectures on Black Holes and Quantum Information.* Princeton Center for Theoretical Science, Princeton University, Princeton NJ 08540 USA, 2015.

[28] V. Balasubramanian (a), S. F. Ross (b), *The dual of nothing*. (a) David Rittenhouse Laboratories, University of Pennsylvania, Philadelphia, PA 19103, USA, (b) Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, South Road, Durham DH1 3LE, U.K., 2002.

[29] T. Hartman, *Lectures on Quantum Gravity and Black Holes*. Cornell University, 2015.

[30] M. Bañados (a), C. Teitelboim (a),(b), J. Zanelli (a), *The Black Hole in Three Dimensional Spacetime*. (a)Centro de Estudios Cient´ıficos de Santiago, Casilla 16443, Santiago 9, Chile, and Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile (b)Institute for Advanced Study, Olden Lane, Princeton, New Jersey 08540, USA, 2001.

[31] P. Calabrese (a), J. Cardy (b),(c), *Entanglement Entropy and Quantum Field Theory*. (a) Rudolf Peierls Centre for Theoretical Physics, (b) Keble Road, Oxford OX1 3NP, U.K. (c) All Souls College, Oxford, 2008.

[32] S. W. Hawking, *Particle Creation by Black Holes*. Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England, 1975.

[33] A. Almheiri (a), N. Engelhardt (b),(c), D. Marolf (d), H. Maxwell (d), *The Entropy of Bulk Quantum Fields and The Entanglement Wedge of an Evaporating Black Hole.* (a) Institute for Advanced Study, Princeton, NJ 08540, USA (b) Department of Physics, Princeton University, Princeton, NJ 08544, USA (c) Gravity Initiative, Princeton University, Princeton NJ 08544, USA (d) Physics Department, University of California, Santa Barbara, CA 93016, 2019.

[34] A. Almheiri (a), R. Mahajan (1),(2), J. Maldacena (1), Y. Zhao (1), *The Page Curve of Hawking Radiation from Semiclassical Gravity*. (1) Institute for Advanced Study, Princeton, NJ 08540, USA (2) Jadwin Hall, Princeton University, Princeton, NJ 08540, USA, 2019.

[35] N. Engelhardt, A. Wall, *Quantum Extremal Surfaces: Holographic Entanglement Entropy beyond the Classical Regime*. Department of Physics University of California, Santa Barbara, Santa Barbara, CA 93106, USA, 2015.

[36] V. Hubeny (a), M. Rangamani (a), T. Takayanagi (b), *A Covariant Holographic Entanglement Entropy Proposal*. (a) Centre for Particle Theory and Department of Mathematical Sciences, Science Laboratories, South Road, Durham DH1 3LE, United Kingdom. (b) Department of Physics, Kyoto University, Kyoto, 606-8502, Japan. 2012.

[37] T. Faulkner (a), A. Lewkowycz (b), J. Maldacena (a), *Quantum corrections to holographic entanglement entropy*. (a) Institute for Advanced Study, Princeton, NJ 08540, USA (b) Jadwin Hall, Princeton University, Princeton, NJ 08544, USA, 2013.

[38] R. Penrose, *Asymptotic Properties of Fields and Space-Times*. Department of Mathemetics, King's College, London, England, 1962.

[39] P. Freund, M. Rubin, *Dynamics of Dimensional Reduction*. Enrico Fermi Institute and Department o f Physics, The University o f Chicago, Chicago, IL 60637, USA, 1980.

[40] S. Hawking, *The path integral approach to quantum gravity*. [in book] General Relativity: An Einstein Centenary Survey. p746-789, 1980.

[41] E. Lieb, M. Ruskai, *Proof of the Strong Subadditivity of Quantum-Mechanical Entropy*. Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1973, tome 19. ≪ Conférences de : P. Lelong, A. Lichnerowicz, E. Lieb et C. Stanojevic ≫, , exp. no 5, p. 36-55, 1973.

[42] A. Almheiri (a), T. Hartman (b), J. Maldacena (a), Edgar Shaghoulian (B), A. Tajdini (b), *Replica Wormholes and the Entropy of Hawking Radiation*. (a) Institute for Advanced Study, Princeton, New Jersey, USA (b) Department of Physics, Cornell University, Ithaca, New York, USA, 2020.

[43] G. Penington, S. Shenker, D. Stanford, Z. Yang, *Replica wormholes and the black hole interior*. Stanford Institute for Theoretical Physics, Stanford University, Stanford, CA 94305, 2020.

[44] L. Sokolowski, *The bizarre anti-de Sitter spacetime*. Astronomical Observatory, Jagiellonian University, Orla 171, Krak´ow 30-244, Poland, 2016.

[45] S. Gubser, *AdS/CFT and gravity*. Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA, 2000.

[46] L. Susskind, *ER=EPR, GHZ, and the Consistency of Quantum Measurements*. Stanford Institute for Theoretical Physics and Department of Physics, Stanford University, Stanford, CA 94305-4060, USA; 2014.

[47] J. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*. Department of Physics, University of California, Santa Barbara, CA 93106, USA, 1989.

[48] E. Witten, *Anti de Sitter Space and Holography.* School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA, 1998/

[49] G. Penington, *Entanglement wedge reconstruction and the information paradox.* Stanford Institute for Theoretical Physics, Stanford University, Stanford CA 94305 USA, 2020.